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On the Trace Embedding and Compact Properties of Finite Element Spaces*

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I. Introduction

In paper [1], the embedding theorem and the compact theorem of Rellich and Kondrachev for the Sobolev spaces are generalized to finite element spaces with certain properties. By means of them, the convergence of finite element methods for a class of nonlinear problems are proved in papers [2-5]. In this paper, the trace embedding and compact theorems for Sobolev spaces will be generalized to the finite element spaces in some conditions. The results will be useful for discussing the convergence of finite element methods for nonlinear problems, especially for the problems with nonlinear boundary conditions.

Let Ω be a bounded polyhedroid domain in R^n. Denote by $a=(a_1,\cdots,a_n)$ the vector with each component a non-negative integer, i.e., a is a multi-index. Define $|a|=\sum_{i=1}^{n} a_i$. For $m\geq 0$ and $p\in[1,\infty]$, $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$ are the general norm and semi-norm of Sobolev space $W^{m,p}(\Omega)$. $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in norm $\|\cdot\|_{m,p,\Omega}$.

Define $L^{m,p}(\Omega)=\{u|u=(u^a)_{|a|\leq m},\ u^a\in L^p(\Omega),|a|\leq m\}$. For $u\in L^{m,p}(\Omega)$,

$$\|u\|_{m,p,\Omega}=\Big[\sum_{|a|\leq m}\int_\Omega |u^a|^p dx\Big]^{\frac{1}{p}},\quad |u|_{m,p,\Omega}=\Big[\sum_{|a|=m}\int_\Omega |u^a|^p dx\Big]^{\frac{1}{p}}$$

if $p<\infty$,

$$\|u\|_{m,\infty,\Omega}=\max_{|a|\leq m} esssup_{x\in\Omega}|u^a(x)| \quad\text{and}\quad |u|_{m,\infty,\Omega}=\max_{|a|=m} esssup_{x\in\Omega}|u^a(x)|$$

if $p=\infty$.

By correspondence $w\in W^{m,p}(\Omega)\to(D^a w)\in L^{m,p}(\Omega)$, $W^{m,p}(\Omega)$ is mapped onto a close subspaces of $L^{m,p}(\Omega)$. We also use $W^{m,p}(\Omega)$ denoting the subspace.

For $h\in(0,1)$, let K_h be a finite subdivision of Ω. Suppose

1) For every $K\in K_h$, it is a close n-simplex or n-paralleltope; $\overline{\Omega}=\bigcup_{K\in K_h} K$ for every h.

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2) For every two elements in K_h, the intersection of them is a empty set or a common face of them.

3) For every $K \in K_h$, the diameter of the smallest ball containing K, denoted by h_K, is not greater than h, while the diameter of the largest ball contained in K, denoted by ρ_K, is not smaller than ηh, with η a positive constant independent of K and h.

For set $S \subset R^n$, let $K_h(S) = \{K \mid K \cap S$ is not empty, $K \in K_h\}$. For $k = 1, \cdots, n-1$, Ω^k is the k-dimensional domain that is the intersection of Ω and some k-dimensional hyperplane in R^n. We also use K to denote the interior point set of K. Define $\Omega_h = \bigcap_{K \in K_h} \mathring{K}$.

2. The Embedding and Compact Properties

Let $\{W_h^m\}$ be a sequence of finite dimensional subspaces in $L^{m,p}(\Omega)$. Suppose that for $\forall\, w_h \in W_h^m$, w_h^α is a polynomial on K, with degree not greater than r, provided $K \in K_h$ and $|\alpha| \leqslant m$. Here r is a positive integer independent of K and h.

Sequence $\{W_h^m\}$ is compatible, if there exists a constant C independent of h, such that,

$$0 \leqslant j < m, \quad \sum_{|\alpha| = j} |w_h^\alpha|_{1,\infty,K} \leqslant C \sum_{|\alpha| = j+1}^{m} h^{|\alpha| - j - 1} \|w_h^\alpha\|_{0,\infty,K}, \quad w_h \in W_h^m \tag{1}$$

are true for $\forall\, K \in K_h$ and $h \in (0,1)$, and the following inequalities

$$0 \leqslant |\alpha| < m, \ |(w_h^\alpha)^{K'}(x) - (w_h^\alpha)^{K''}(x)| \leqslant C \sum_{|\beta| = |\alpha| + 1}^{m} h^{|\beta| - |\alpha|} \sup_{K \in K_h(x)} |w_h^\beta|_{0,\infty,K}, \tag{2}$$

are true for $\forall\, K', K'' \in K_h(x)$, $\forall\, x \in \Omega$, $\forall\, w_h \in W_h^m$ and $h \in (0,1)$. Here $(w_h^\alpha)^K$ is the continuous extension of $(w_h^\alpha)|_{\mathring{K}}$ to K.

Let W_0 be a subspace of $L^{m,p}(\Omega)$. $\{W_h^m, W_0\}$ is weekly closed if for $w_k \in W_{h_k}^m (k = 1, 2, \cdots)$, w_∞ is in W_0 when w_k weekly converges to w_∞ in $L^{m,p}(\Omega)$ and h_k converges to 0.

For the case of $m = 1$ and $m = n = 2$. we give the generalized trace embedding and compact theorems.

Theorem 1 Let $m = 1$ or $m = n = 2$, suppose $\{W_h^m\}$ is compatible and $p \in (1, \infty)$. If $0 \leqslant j < m$ and $(m - j)p < n$, then for positive integer k and real q_j, there exists a constant C independent of h, such that,

$$\forall\, w_h \in W_h^m, \quad \sum_{|\alpha| = j} \|w_h^\alpha\|_{0,q_j,\Omega^k} \leqslant C \|w_h\|_{m,p,\Omega} \tag{3}$$

are true for $\forall\, h \in (0,1)$, provided $n - (m-j)p < k < n$ and $p \leqslant q_j \leqslant kp/(n - (m-j)p)$. If $(m-j)p = n$, then for $\forall\, k \in \{1, \cdots, n-1\}$ and $q_j \in [p, \infty)$ there exists a constant C independent of h such that inequalities (3) are true for $h \in (0,1)$.

Theorem 2 Let $m = 1$ or $m = n = 2$, suppose $\{W_h^m\}$ is compatible and $p \in (1, \infty)$.

For $j\in\{0,1,\cdots,m-1\}$, if integer k and real q_j satisfy that $n-(m-j)p<k<n$ and $p\leqslant q_j<kp/(n-(m-j)p)$ when $n>(m-j)p$, $1\leqslant k\leqslant n-1$ and $p\leqslant q_j<\infty$ when $n=(m-j)p$, then following two conclusions are true.

1) If $w_\tau\in W_{h_\tau}^m$ $(\tau=1,2,\cdots)$ and w_τ weekly converges to 0 in $L^{m,p}(\Omega)$ and h_τ converges to 0, then $\lim\limits_{\tau\to\infty}\sum\limits_{|a|=j}\|w_\tau^a\|_{0,q_j,\Omega^k}=0$.

2) Suppose that $\{W_h^m, W^{m,p}(\Omega)\}$(or$\{W_h^m, W_0^{m,p}(\Omega)\}$) is weekly closed and has the approximability that for $w\in W^{m,p}(\Omega)$ (or $W_0^{m,p}(\Omega)$),

$$\lim\inf_{w_h\in W_h^m}(\|w-w_h\|_{m,p,\Omega}+\sum_{|a|=j}\|D^a w-w_h^a\|_{0,q_j,\Omega^k})=0$$

When $w_\tau\in W_{h_\tau}^m$ $(\tau=1,2,\cdots)$ and $\{w_\tau\}$ is weekly converges to $w_\infty\in W^{m,p}(\Omega)$(or $W_0^{m,p}(\Omega)$) and h_τ converges to 0, the following is true,

$$\lim\inf_{w_h\in W_h^m}\sum_{|a|=j}\|D^a w-w_h^a\|_{0,q_j,\Omega^k}=0,$$

For Ω^k, its subset maybe a face of some element $K\in K_h$. On the subset, w_h^a is defined by continuous extension of $w_h^a|_K$ to K. If the element K is not unique, one is choose arbitrarily. For similar case later, it is dealt similarly.

For a class of finite element spaces, such as nonconforming, primal hybrid and quasi-conforming element spaces, the conditions for compatibility, week closedness and approximability are referred to paper [5].

3. The Proof of the Results

We assume, in this section, that $p\in(1,\infty)$ and $\{W_h^m\}$ has the compatibility, and that $1\leqslant\nu<n$, $j\in\{0,1,\cdots,m-1\}$. Denote by $S_h(B)$, for some set $B\subset R^n$, the set of all points with the distance between them and B not greater than h, and by C a positive constant independent of h which may be different in different places. Sometimes also denote it by C_1, C_2, etc. For $w_h\in W_h^m$, $\nabla_\nu w_h^a$ is the gradient of w_h^a on Ω^ν in ν-dimension.

Lemma 1 Let $m=1$ and Ω^ν a ν-hypercube with side equal to 1. If $\nu<p$, $q\geqslant 1$, then for $w_h\in W_h^1$, $x,y\in\Omega^\nu$, the following inequality,

$$|w_h^0(x)-w_h^0(y)|\leqslant C_1\{\|x-y\|^{1-\frac{\nu}{p}}\|\nabla_\nu w_h^0\|_{0,p,\Omega^\nu}+h^{1-\frac{n}{q}}\sum_{|a|=1}\|w_h^a\|_{0,q,S_h(\Omega^\nu)}\}\qquad(4)$$

is true when $\|x-y\|\leqslant h$, and

$$|w_h^0(x)-w_h^0(y)|\leqslant C_2\|x-y\|^{1-\frac{\nu}{p}}\{\|\nabla_\nu w_h^0\|_{0,p,\Omega^\nu}+h^{\frac{\nu-n}{p}}\sum_{|a|=1}\|w_h^a\|_{0,p,S_h(\Omega^\nu)}\}\qquad(5)$$

is true when $\|x-y\|>h$.

Proof For $0<t<1$, Ω_t is the ν-hypercube with the face parallel to the one of Ω^ν and the side equal to t and $\overline{\Omega_t}\subset\Omega^\nu$. If $x,y\in\Omega^\nu$ with $|x-y|=\sigma<1$, then there exists a hypercube Ω_σ, such that $x,y\in\Omega_\sigma\subset\Omega^\nu$. For $z\in\Omega_\sigma$.

$$w_h^0(x)=w_h^0(z)-\int_0^1\frac{d}{dt}w_h^0(x+t(z-x))dt+\sum_s t_s(w_h^0(y_s^+)-w_h^0(y_s^-)),$$

where $y_s=x+t_s(z-x)$ is the discontinuous point of $w_h^0(x+t(z-x))$ for t, $w_h^0(y_s^+)$ and $w_h^0(y_s^-)$ are the left and right limits of it respectively. Therefore,

$$|w_h^0(x)-w_h^0(z)|<\sqrt{\nu}\,\sigma\int_0^1|\nabla_\nu w_h^0(x+t(z-x))|dt+\sum_s|w_h^0(y_s^+)-w_h^0(y_s^-)|,$$

$$|w_h^0(x)-\frac{1}{\sigma^\nu}\int_{\Omega_\sigma}w_h^0(z)dz|<C\sigma^{-\nu+1}\int_{\Omega_\sigma}dz\int_0^1|\nabla_\nu w_h^0(x+t(z-x))|dt$$
$$+\sigma^{-\nu}\int_{\Omega_\sigma}\sum_s|w_h^0(y_s^+)-w_h^0(y_s^-)|dz \qquad (6)$$

By the method used in [6], we have

$$\sigma^{1-\nu}\int_{\Omega_\sigma}dz\int_0^1|\nabla_\nu w_h^0(x+t(z-x))|dt<C\sigma^{1-\frac{\nu}{p}}\|\nabla_\nu w_h^0\|_{0,p,\Omega'} \qquad (7)$$

For $\mu>1$, let μ' the dual number of it. By the inequalities (2) and the method used in [7], the following inequality,

$$\int_{\Omega_\sigma}\sum_s|w_h^0(y_s^+)-w_h^0(y_s^-)|dz<Ch^{1-n/\mu-1/\mu'}(\sigma+h)^{\frac{1}{\mu'}}(\int_{\Omega_\sigma}dz)^{\frac{1}{\mu'}}\times\sum_{|\alpha|=1}(\iint_{\Omega_\sigma S_h(\overline{xz})}|w_h^\alpha|^\mu dydz)^{\frac{1}{\mu}}$$

is true, where $\overline{xz}$ is the segment connecting x and y. Obviously,

$$(\int_{\Omega_\sigma}dz)^{\frac{1}{\mu'}}=\sigma^{\frac{\nu}{\mu'}} \qquad (9)$$

When $\sigma<h$, we have

$$\iint_{\Omega_\sigma S_h(\overline{xz})}|w_h^\alpha|^\mu dydz<\iint_{\Omega_\sigma S_h(\Omega')}|w_h^\alpha|^\mu dydz=\sigma^\nu\|w_h^\alpha\|^\mu_{0,\mu,S_h(\Omega')} \qquad (10)$$

When $\sigma>h$, we have, by exchanging the integration order and the estimate of integrate domain,

$$\iint_{\Omega_\sigma S_h(\overline{xz})}|w_h^\alpha|^\mu dydz<Ch^{-1}\sigma\|w_h^\alpha\|^\mu_{0,\mu,S_h(\Omega')}$$

When $\sigma\leqslant h$, by setting $\mu=p$ and using inequalities (6)—(10), we get

$$|w_h^0(x)-\frac{1}{\sigma^\nu}\int_{\Omega_\sigma}w_h^0(z)dz|<C\{\sigma^{1-\frac{\nu}{p}}\|\nabla_\nu w_h^0\|_{0,p,\Omega'}+h^{1-\frac{n}{q}}\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,q,S_h(\Omega')}\} \qquad (12)$$

When $\sigma>h$, using inequalities (6)—(9) and (11), we get

$$|w_h^0(x)-\frac{1}{\sigma^\nu}\int_{\Omega_\sigma}w_h^0(z)dz|<C\sigma^{1-\frac{\nu}{p}}\{\|\nabla_\nu w_h^0\|_{0,p,\Omega'}+h^{\frac{\nu-n}{p}}\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,p,S_h(\Omega')}\} \qquad (13)$$

Inequalities (12) and (13) are also true when x replaced by y. So it follows that inequalities (4) and (5) are true. When $\|x-y\|>1$, we can choose $\sigma=1$ in the above discussion.

Lemma 2 Let $m=1$ and Ω^ν a ν-hypercube with side equal to 1. If $\nu<p$, $q\geqslant1$, then for $\forall w_h\in W_h^1$, $\forall x\in\Omega^\nu$, the following inequality,

$$|w_h^0(x)| < C\{\sum_{|\alpha|<1}\|D^\alpha w_h^0\|_{0,p,\Omega'}^{1-s} + h^{\frac{\nu-n}{\nu}(1-s)}\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,p,S_h(\Omega')}^{1-s}\} \times \{\|w_h^0\|_{0,q,\Omega'}^s + h^{(1+\frac{\nu-n}{q})s}\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,q,S_h(\Omega')}^s\} \tag{14}$$

is true, where $s=(p-\nu)q/(\nu p+(p-\nu)q)$.

Proof Set $U=\|\nabla_\nu w_h^0\|_{0,p,\Omega'}+h^{\frac{\nu-n}{p}}\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,p,S_h(\Omega')}$. We can assume that $U>0$.

For $\forall y\in\Omega'$, denote $\rho=\|x-y\|$, $\zeta=(\frac{w_h^0(x)}{C_1U})^{p/(p-\nu)}$.

Firstly assume $\zeta\leqslant 1$. When $\rho<\zeta$ we have $|w_h^0(x)|-C_1U\rho^{1-\frac{\nu}{p}}>0$. When $\rho<h$, by inequality (4), we get

$$|w_h^0(y)|+h^{1-\frac{n}{q}}\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,q,S_h(\Omega')}>|w_h^0(x)|-C_1U\rho^{1-\frac{\nu}{p}} \tag{15}$$

When $\rho>h$, from inequality (5) we get

$$|w_h^0(y)|>|w_h^0(x)|-C_1U\rho^{1-\frac{\nu}{p}} \tag{16}$$

Define $B_h=\{z|z\in R^n$ and $\|z-x\|<h\}$. The both sides of inequality (15) are taken q-power and integrated on $B_h\cap\Omega'$, and the both sides of inequality (16) are taken q-power and integrated on $\Omega'-B_h$, then we get two new inequalities. From the inequalities we get,

$$\int_{\Omega'}|w_h^0(y)|^q\mathrm{d}y+C_2h^{\nu+(1-\frac{n}{q})q}\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,q,S_h(\Omega')}^q>C_3|w_h^0(x)|^{q+\nu p/(p-\nu)}U^{-\nu p/(p-\nu)}$$

It follows that inequality (14) is true.

Secondly let $\zeta>1$, then $|w_h^0(x)|-C_1U\rho^{1-\frac{\nu}{p}}>|w_h^0(x)|-|w_h^0(x)|\rho^{1-\frac{\nu}{p}}>0$ when $\rho<1$. Assume $t>0$. Let $q=t$ in inequality (15), and the both sides are taken t-power and integrated on $B_h\cap\Omega'$, and the both sides of inequality (16) are taken t-power and integrated on $\Omega'-B_h$. It can be deduced that

$$\int_{\Omega'}|w_h^0(y)|^t\mathrm{d}y+C_2h^{\nu+(1-\frac{n}{q})t}\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,q,S_h(\Omega')}^t>C_4|w_h^0(x)|^t$$

Setting $t=\dfrac{(p-\nu)q+\nu p}{p}$, by Holder inequality, we get

$$|w_h^0(x)|^{\frac{(p-\nu)q+\nu p}{\nu}}<C\{\|w_h^0\|_{0,q,\Omega'}^{q(p-\nu)/p}\|w_h^0\|_{0,p,\Omega'}^{\nu} + h^{\nu+(1-\frac{n}{t})t}\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,q,S_h(\Omega')}^{q(p-\nu)/p}\|w_h^\alpha\|_{0,p,S_h(\Omega')}^{\nu}\}$$

Noting that

$$\nu+(1-\frac{n}{t})t>(1+\frac{\nu-n}{q})q(p-\nu)/p+(\nu-n)\nu/p \text{ and } \|w_h^0\|_{0,p,\Omega'}<\sum_{|\alpha|<1}\|D^\alpha w_h^0\|_{0,p,\Omega'},$$

we show inequality (14).

Proof of Theorem 1

i) Assume that $m=1$ and $n>p$. Not losing the general case, we suppose that Ω is a sum aggregate of some hypercubes with their sides equal to 2 and paraller to the coordinate axis (see lemma 5.19 in [6]).

Let R_0^k a k-dimensional coordinate subspace in R^n. There exists a one to one project Ω_0^k of Ω^k on R_0^k. Set ν the largest of all integers smaller than p. Then $p>\nu>0$, and $n-\nu<k$ because $k>n-p$. Define $\mu=\binom{k}{n-\nu}$, and denote by $E_i(1\leqslant i\leqslant\mu)$ the all coordinate spaces of $(n-\nu)$-dimension. Let Ω_i the project of Ω_0^k (so Ω^k) on E_i, $\Omega_{i,x}$ the intersection of Ω and the ν-dimensional hyperplane which is orthogonal to E_i and through x for $x\in\Omega_i$. Hence $\Omega_{i,x}$ contains a ν-hypercube with sides equal to 1 and x a vertex. For $q=np/(n-p)$ and $w_h\in W_h^1$, by lemma 2, we get

$$\sup_{y\in\Omega_{i,x}}|w_h^0(y)|^{(n-\nu)p/(n-p)}\leqslant C\{\|w_h^0\|_{0,p,\Omega_{i,x}}^{(p-\nu)q/p}+h^{(1+\frac{\nu-n}{q})(p-\nu)q/p}\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,q,S_h(\Omega_{i,x})}^{(p-\nu)q/p}\}$$
$$\times\{\sum_{|\alpha|<1}\|D^\alpha w_h^0\|_{0,p,\Omega_{i,x}}^{\nu}+h^{\frac{\nu-n}{p}\nu}\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,p,S_h(\Omega_{i,x})}^{\nu}\}\tag{17}$$

Let dx^i and dx_*^i be the elements of the integral of E_i and the orthogonal complement of E_i respectively. Integrating inequality (17) on Ω_i and using Holder inequality we have

$$\int_{\Omega_i}\sup_{y\in\Omega_{i,x}}|w_h^0(y)|^{(n-\nu)p/(n-p)}dx^i$$
$$\leqslant C\{[\iint_{\Omega_i\Omega_{i,x}}\sum_{|\alpha|<1}|D^\alpha w_h^0|^p dx_*^i dx^i]^{\nu/p}+h^{(\nu-n)\nu/p}\sum_{|\alpha|=1}[\int_{\Omega_i}\int_{S_h(\Omega_{i,x})}|w_h^\alpha(y)|^p dy dx^i]^{\nu/p}\}\times$$
$$\{[\iint_{\Omega_i\Omega_{i,x}}|w_h^0(x)|^q dx_*^i dx^i]^{\frac{p-\nu}{p}}+h^{(1+\frac{\nu-n}{q})(p-\nu)q/p}\sum_{|\alpha|=1}[\int_{\Omega_i}\int_{S_h(\Omega_{i,x})}|w_h^\alpha(y)|^q dy dx^i]^{\frac{p-\nu}{p}}\}$$

Since the following inequalities,

$$\int_{\Omega_i}\int_{S_h(\Omega_{i,x})}|w_h^\alpha(y)|^q dy dx^i\leqslant Ch^{n-\nu}\|w_h^\alpha\|_{0,q,\Omega}^q,$$
$$\int_{\Omega_i}\int_{S_h(\Omega_{i,x})}|w_h^\alpha(y)|^p dy dx^i\leqslant Ch^{n-\nu}\|w_h^\alpha\|_{0,p,\Omega}^p$$

are true, we deduce that

$$\int_{\Omega_i}\sup_{y\in\Omega_{i,x}}|w_h^0(y)|^{(n-\nu)p/(n-p)}dx^i\leqslant C\{\sum_{|\alpha|<1}\|D^\alpha w_h^0\|_{0,p,\Omega}^{\nu}+\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,p,\Omega}^{\nu}\}$$
$$\times\{\|w_h^0\|_{0,q,\Omega}^{(p-\nu)q/p}+h^{(p-\nu)q/p}\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,q,\Omega}^{(p-\nu)q/p}\}.\tag{18}$$

Let $dx^{(k)}$ the element of the integral on R_0^k and $q_j=kp/(n-p)$. Applying lemma 5.9 in [6] with $\lambda=\binom{k-1}{n-\nu-1}$, we get

$$\|w_h^0\|_{0,q_j,\Omega^k}^{q_j}\leqslant C\int_{\Omega_0^k}\prod_{i=1}^{\mu}\sup_{y\in\Omega_{i,x}}|w_h^0(y)|^{q_j/\mu}dx^{(k)}\leqslant C\prod_{i=1}^{\mu}\{\int_{\Omega_i}\sup_{y\in\Omega_{i,x}}|w_h^0(y)|^{q_j\lambda/\mu}dx^i\}^{1/\lambda}\tag{19}$$

Noting $q_j\lambda/\mu=(n-\nu)p/(n-p)$ and inequalities (18) and (19), we get

$$\|w_h^0\|_{0,q_j,\Omega^k}\leqslant C\{\|w_h^0\|_{0,q,\Omega}^{(p-\nu)q/p}+h^{(p-\nu)q/p}\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,q,\Omega}^{(p-\nu)q/p}\}^{\frac{\mu}{q_j\lambda}}$$
$$\{\sum_{|\alpha|<1}\|D^\alpha w_h^0\|_{0,p,\Omega}^{\nu}+\sum_{|\alpha|=1}\|w_h^\alpha\|_{0,p,\Omega}^{\nu}\}^{\frac{\mu}{q_j\lambda}}\tag{20}$$

By the technique of affine transformation, we can easily proved that the following inequalities,

$$\forall v \in P_r(K),\ |v|_{0,s,K} \leqslant C h^{\frac{n}{s}-\frac{n}{t}} |v|_{0,t,K},\quad K \in K_h \tag{21}$$

are true for $\forall\, s, t \in [1, \infty]$. From inequalities (1) and (21) it is deduced that

$$\sum_{|\alpha|=1} \| D^\alpha w_h^0 \|_{0,p,\Omega} \leqslant C \sum_{|\alpha|=1} \| w_h^\alpha \|_{0,p,\Omega} \tag{22}$$

$$\| w_h^\alpha \|_{0,q,\Omega} \leqslant C h^{\frac{n}{q}-\frac{n}{p}} \| w_h^\alpha \|_{0,p,\Omega} = C h^{-1} \| w_h^\alpha \|_{0,p,\Omega} \tag{23}$$

From theorem 1 in [1] or theorem 2.2.1 in [5], we know

$$\| w_h^0 \|_{0,q,\Omega} \leqslant C \| w_h \|_{1,p,\Omega} \tag{24}$$

Substituting inequalities (23) and (24) into inequalities (20), inequalities (3) is proved in the case of $q_j = kp/(n-p)$. Hence it is also true for other $q_j < kp/(n-p)$.

ii) Assume that $m=1$ and $n=p$. For $\forall\, k \in \{1, 2, \cdots, n-1\}$, $q_j \in [p, \infty)$, it is always possible to select σ such that $1 < \sigma < p$, $n-\sigma < k$ and $q_j \leqslant k\sigma/(n-\sigma)$. From step i) we know,

$$\| w_h^0 \|_{0,q_j,\Omega^k} \leqslant C \| w_h \|_{1,\sigma,\Omega} \leqslant C \| w_h \|_{1,p,\Omega}$$

So far theorem 1 is proved for the case of $m=1$.

iii) For the case of $m=n=2$, it is necessary to consider the case of $j=1$ and $p \leqslant n$. The result can be got by similar way above. Theorem 1 is proved to be true.

Proof of Theorem 2

1) Assume that $w_\tau \in W_{h_\tau}^m$ for $\tau = 1, 2, \cdots$, and that $\{w_\tau\}$ is weekly convergent to 0 in $L^{m,p}(\Omega)$ and h_τ converges to 0. First, we consider the case of $m=1$ and $n > p$. Let k, q_j satisfy that $n-p < k < n$, $p \leqslant q_j < kp/(n-p)$. We can choose σ with $n-\sigma < k$, $q_j < k\sigma/(n-\sigma)$ and $\sigma > p$. Set $q = n\sigma/(n-\sigma)$, then $q < np/(n-p)$. Replacing p by σ and q_j by $r_j \equiv k\sigma/(n-\sigma)$ in inequalily (20), we get

$$\begin{aligned} \| w_r^0 \|_{0,r_j,\Omega^k} \leqslant C \{ \| w_\tau^0 \|_{0,q,\Omega}^{(\sigma-\nu)q/\sigma} + h^{(\sigma-\nu)q/\sigma} \sum_{|\alpha|=1} \| w_\tau^\alpha \|_{0,q,\Omega}^{(\sigma-\nu)q/\sigma} \}^{\frac{\mu}{r_j\lambda}} \\ \times \{ \sum_{|\alpha|<1} \| D^\alpha w_\tau^0 \|_{0,\sigma,\Omega}^\nu + \sum_{|\alpha|=1} \| w_\tau^\alpha \|_{0,\sigma,\Omega}^\nu \}^{\frac{\mu}{r_j\lambda}} \end{aligned} \tag{25}$$

From theorem 2 in [1] or theorem 2.2.1 in [5] we know

$$\lim \| w_\tau^0 \|_{0,q,\Omega} = 0. \tag{20}$$

By inequalities (21) and $1 + \frac{n}{q} - \frac{n}{p} = \frac{n(p-\sigma)}{\sigma p} > 0$ we have

$$h_\tau \sum_{|\alpha|=1} \| w_\tau^\alpha \|_{0,q,\Omega} \leqslant C h_\tau^{1+\frac{n}{q}-\frac{n}{p}} \sum_{|\alpha|=1} \| w_\tau^\alpha \|_{0,p,\Omega} \to 0 \ \ as\ \ \tau \to \infty. \tag{27}$$

It follows that $\| w_\tau^0 \|_{0,r_j,\Omega^k} \to 0$. The conclusion 1) in theorem 2 is true in the case of $m=1$ and $n > p$.

If $m=1$ and $n=p$, we choose $\sigma<p$ such that $n-\sigma<k$ and $q_j<k\sigma/(n-\sigma)$. Using the above way, we can get the result. For the case of $m=n=2$, the conclusion 1) of theorem can be shown similarly.

2) Assume that $w_\tau \in W^m_{h_\tau}$ for $\tau=1,2,\cdots$, and that w_τ is weekly convergent to $w_\infty \in W^{m,p}(\Omega)$ (or $W^{m,p}_0(\Omega)$) and h_τ converges to 0, then there exist, by approximability, $v_\tau \in W^m_{h_\tau}$ such that

$$\lim(\| w_\infty - v_\tau \|_{m,p,\Omega} + \sum_{|\alpha|=j} \| D^\alpha w_\infty - v^\alpha_\tau \|_{0,q_j,\Omega^k}) = 0.$$

So $w_\tau - v_\tau$ is weekly convergent to 0, The conclusion 2) is got from the above equality, conclusion 2) and triangle inequality. Theorem 2 is proved.

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