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A Pair of Formulae Generated by Lagrange's Expansion

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Let Γ denote the ring of formal power series over the complex field $\mathbf{C}$. Let $\phi\in\Gamma$ with $\varphi(0)=0$ and $\varphi'(0)\neq 0$ so that it has a compositional inverse ψ with $\psi(\phi(t))=\varphi(\psi(t))=t$. Then for every $f\in\Gamma$ and $a\in\mathbf{C}$ with $a\neq 0$ we have Lagrange's formal expansion

$$(f(\psi(t)))^a=(f(0))^a+\sum_{n\geq 1}\frac{t^n}{n!}D_0^{n-1}\{D(f(t))^a\cdot(t/\varphi(t))^n\}, \tag{1}$$

where $D\equiv d/dt$ and D_0 means formal differentiation at $t=0$.

Theorem 1. Denote $\left\{ {a \atop 0} \right\}:=(f(0))^a$ and for $n\geq 1$ define

$$\left\{ {a \atop n} \right\}:=\frac{1}{n!}D_0^{n-1}(Df^a(t)\cdot(t/\varphi(t))^n). \tag{2}$$

Then for $(a,\beta)\in\mathbf{C}\times\mathbf{C}$ we have a convolution formula of the form

$$\left\{ {a+\beta \atop n} \right\}=\sum_{k=0}^{n}\left\{ {a \atop k} \right\}\left\{ {\beta \atop n-k} \right\}. \tag{3}$$

Theorem 2. Let $\sigma(n)$ denote the set of partitions of n, usually repressented by $(k)\equiv(k_1,\cdots,k_n)$: $=1^{k_1}2^{k_2}\cdots n^{k_n}$ with $k_1+2k_2+\cdots+nk_n=n$. Let $\phi(t)$ and $f(t)$ b be defined as in (1) with $f(0)=1$. Then we have

$$\left\{ {a\beta \atop n} \right\}=\sum_{\sigma(n)}\binom{a}{(k)}\prod_{i=1}^{n}\left\{ {\beta \atop i} \right\}^{k_i} \tag{4}$$

where the summation is taken over all the partitions of n, and $\binom{a}{(k)}$ denotes the multinomial coefficient with $(k)\equiv(k_1,\cdots,k_n)$.

Both (3) and (4) imply various special identities for special number sequ ences and special polynomials. For instance, taking $\varphi(t)=t(1+t)^{-b}$, $f(t)=1+t$, one easily obtains

$$A_n(a+\beta,\ b)=\sum_{0\leq k\leq n}A_k(a,b)A_{n-k}(\beta,b), \tag{5}$$

$$A_n(a\beta,\ b)=\sum_{\sigma(n)}\binom{a}{(k)}\prod_{i=1}^{n}A_i(\beta,\ b)^{k_i} \tag{6}$$

where $A_n(x,\ b)=\frac{x}{x+bn}\binom{x+bn}{n}$. (5) is the well known identity due to Hagen and Rothe, and (6) gives an identity of Chu Wenchang for $b=0$.

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