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Sublinear and Homogeneous Quasiconcavity (Quasiconvexity) Properties of the Optimal Value Function in Parametric Nonlinear Programming*

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Abstract

In this paper, sublinear and homogeneous quasiconcavity (quasiconvexity) properties of the optimal value function in parametric nonlinear programming problem:

$$P(u): \begin{cases} \max f(x, u) \\ \text{s.t. } x \in C(u) \end{cases}$$

is systematically researched. The homogeneous concavity and convexity properties in [1] are generalized by many new results of our paper.

I. Introduction

The general parametric nonlinear programming is studied in this paper. Meanwhile, several parametric optimization problems the we may further specialize $P(u)$ are researched:

$$P_1(u): \begin{cases} \max f(x, u) \\ \text{s.t. } x \in C(u) = \{x \in \Omega \mid g(x, u) \leqslant 0,\ h(x, u) = 0\} \end{cases}$$

$$P_2(u): \begin{cases} \max f(x, u) \\ \text{s.t. } x \in C(u) = \{x \in \Omega \mid g_i(x) \leqslant u_i,\ h_j(x) = u_j,\ i = 1, 2, \cdots, p,\ j = p+1, m\} \end{cases}$$

$$P_3(u): \begin{cases} \max f(x, u) \\ \text{s.t. } x \in \Omega \end{cases} \qquad P'(u): \begin{cases} \max f(x) \\ x \in C(u) \end{cases}$$

Generally, optimal value function is defined by

$$f^*(u) = \begin{cases} \sup\{f(x, u) \mid x \in C(u)\} & C(u) \neq \varnothing \\ -\infty & C(u) = \varnothing \end{cases}$$

where $f(x, u)$: $R^n \times R^n \to R$, $f(x)$: $R^n \to R$, $g \in R^p$, $h \in R^{m-p}$, r^m is regarded as the parametric space, and C is a set value mapping from R^m to R^n, as well as several

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specializations of this problem. Its graph is defined by $G(C)=\{(x,u)\mid x\in C(u)\}$.

Main topics of this paper is: sublinear property of the optimal value function is considered in section one. The Second section concerns the homogeneous quasiconcavity and quasiconvexity properties of the optimal value function. Many results of the paper are valid or general abstract spaces, for example, vector, vector toplogical, vector normed, or Banach spaces.some results are applicable to the optimal control problem and mathematical economics, etc. This will be the topic of a forth coming paper.

2. ar property of the optimal value function

Definition 2.1 Function $f: R^n \to R$ is called sublinear, if it satisfies the following conditions:

(a) $f(ax)=af(x)$, for every $x\in R^n$, $a>0$

(b) $f(x_1+x_2)<f(x_1)+f(x_2)$, for all $x_1, x_2\in K$, where K is a vector subspace in R^n.

For the above notion, it is not difficult to generlize the case of set value mapping. We call set value mapping C sublinear on a linear subspace U, if:

(a) $C(au)=aC(u)$ for every $u\in U$, $a>0$.

(b) $C(u_1+u_2)\subset C(u_1)+C(u_2)$ for all $u_1, u_2\in U$.

Note if f which is fintie is positively homogeneous and zero belongs to set Ω, then we have $f(0)=0$. Also, the set value mapping C is homogeneous equivalently that if set $G(C)\cap(R^n\times U)$ is a cone and the above definition contain relation is substituted by equal-sign. If mapping C is linear, then its equivalent form: $G(c)\cap(R^n\times U)$ is linear subspace (see Berge [2]).

Theorem 2.1 Consider the parametric programming problem $P(u)$. If f is sublinear in set $X\subset R^n\times P(R^m)$, where superspace $R^n\times P(R^m)$, C is sublinear in U. X is linear subspace of superspacg. U is a linear subspace of R^m, then the optimal value function f'' is sublinear on U.

Proof At first, let $a>0$, for every $u\in U$,

$$f''(au)=\sup_{x\in C(au)} f(x,au)=\sup_{x\in C(u)} f(ax,au)=a\sup_{x\in C(u)} f(x,u)=af''(u)$$

Second, we only prove:

$$f''(au_1+\beta u_2)<af(u_1)+\beta f(u_2)$$

In fact, for every $a>0$, $\beta>0$, and for all $u_1, u_2\in U$, because U is linear subspace, $au_1+\beta u_2\in U$, By the sublinear property of f and C,

$$\begin{aligned} f''(au_1+\beta u_2) &= \sup_{x\in C(au_1+\beta u_2)} f(x, au_1+\beta u_2) \\ &< \sup_{\substack{x_1\in C(u_1)\\ x_2\in C(u_2)}} f(ax_1+ax_2, au_1+\beta u_2) \end{aligned}$$

$$\leqslant \sup_{\substack{x_1 \in C(u_1) \\ x_2 \in C(u_2)}} [f(ax_1 + ax_2 + f(au_1 + f(au_1 + \beta u_2))]$$

$$\leqslant a \sup_{x_1 \in C(u_1)} f(x_1, u_1) + \beta \sup_{x_2 \in C(u_2)} f(x_2, u_2) = af^n(u_1) + \beta f^n(u_2)$$

This shows that f^n is sublinear.

Theorem 2.2 For parametric nonlinear programming $P_3(u)$. If for every $x \in \Omega$, f is sublinear in U, Ω and U are both linear subspaces. Then f^n is sublinear on U.

Proof For all $u_1, u_2 \in U$, $x \in \Omega$, and $a, \beta > 0$; by the sublinear property of f:

$$f^n(au_1 + \beta u_2) = \sup_{x \in \Omega} f(x, au_1 + \beta u_2)$$

$$\leqslant \sup_{x \in \Omega} [f(x, au_1)) + \sup_{x \in \Omega} f(x, \beta u_2)]$$

$$\leqslant \sup_{x \in \Omega} f(x, au_1) + \sup_{x \in \Omega} f(x, \beta u_2)$$

$$= a \sup_{x \in \Omega} f(x, u_1) + \beta \sup_{x \in \Omega} f(x, u_2) = aF^n(u_1) + \beta f^n(u_2)$$

Remark 2.1 This result can also be extended to the parametric optimization $P'(u)$, but it is rather difficult to obtain the similar results of Theorem 2.1 and Theorem 2.2 to problems $P_1(u)$ and $P_2(u)$.

Remark 2.2 Even the conditions of Theorem 2.1—2.2 are strengthened, i. e., the objective function f and the constraint set value mapping C are all linear, only the sublinear can be obtained.

For the further discussion, we introduce the notion of contraction set.

Definition 2.2 Let A is a set, if for every $a > 0$, $aA \subset A$, then we call set A contraction set. Set value mapping C is contract, if $C(au) \subset C(u)$.

Theorem 2.3 Consider problem $P(u)$, if f is sublinear in U for every $x \in \Omega$ constraint set value map C is contract and sublinear for every $u \in U$, Ω and U are all vector subspaces, then f^n is sublinear on U.

Proof For all $u_1, u_2 \in U$, $a, \beta \in R_+$, by sublinearness of f and contraction and sublinearness of C:

$$f^n(au_1 + \beta u_2) = \sup_{x \in C(au_1 + \beta u_2)} f(x, au_1 + \beta u_2)$$

$$\leqslant \sup_{\substack{x_1 \in C(au_1) \\ x_2 \in C(\beta u_2)}} f(x, au_1 + \beta u_2)$$

$$\leqslant \sup_{\substack{x_1 \in C(au_1) \\ x_2 \in C(\beta u_2)}} [f(x, au_1) + f(x, \beta(u_2))]$$

$$\leqslant a \sup_{x \in C(au_1)} f(x, u_1) + \beta \sup_{x \in C(\beta u_2)} f(x, u_2)$$

$$= af^n(u_1) + \beta f^n(u_2).$$

We call C fixed mapping, if for every $u \in U$, $a>0$, $C(au) \subset C(u)$, where U is a cone. It is clear that a fixed mapping is certainly contract.

Lemma 2.1 If g and h are positively homogeneous in U for every x, then the constraint set mapping C is fixed mapping on U.

Proof For every $a>0$ and $u \in U$ because

$$C(u) = \{x \in \Omega \mid g(x, u) \leqslant 0,\ h(x, u) = 0\}.$$

We may prove by simple calculation:

$$\begin{aligned} C(au) &= \{x \in \Omega \mid g(x, au) \leqslant 0,\ h(x, au) = 0\} \\ &= \{x \in \Omega \mid a g(x, u) \leqslant 0,\ a h(x, u) = 0\} \\ &= \{x \in \Omega \mid g(x, u) \leqslant 0,\ h(x, u) = 0\} = C(u). \end{aligned}$$

For the applications of lemma 2.1 and Theorem 2.1 we obtain the following result.

Thoerem 2.4 For $P(u)$. If f is sublinear in u for every $x \in \Omega$, g and h are both positively homogeneous in U for every $x \in \Omega$, Ω and U are all vector subspaces, then $f''(u)$ is also subadditive.

Remark 2.3 Theorem 2.4 is not applicable for problems $P_1(u)$ and $P_2(u)$.

Theorem 2.5 Consider the parametric optimization problem $P(u)$. If f is sublinear in x for every $x \in C(u)$, and constraint set value mapping C is fixed and subadditive, then the optimal value function f'' is surely sublinear on U.

3. Homogeneous quasiconcavity and quasiconvexity properties of the optimal value function

Definition 3.1 $\varphi: R^n \to R$ is quasiconvex on its domain Ω, if for all $x_1, x_2 \in \Omega$, $a, \beta > 0$ $(a + \beta = 1)$ We have:

$$\varphi(ax_1 + \beta x_2) \leqslant \max\{\varphi(x_1),\ \varphi(x_2)\}.$$

If $(-\varphi)$ is quasiconvex, we call φ quasiconcave, meanwhile, φ is called homogeneous quasiconvex or quasiconcave, if φ is positively homogeneous.

It is very easy to extend the above notions to set value mapping.

Definition 3.2 Suppose the set value mapping $C: U \to P(R^n)$ (superspace in R^n) is concave if for all $u_1, u_2 \in U$, $a, \beta > 0$ $(a + \beta = 1)$

$$C(au_1 + \beta u_2) \subset ac(u_1) + \beta c(u_2)$$

where U convex. If the converse contain relation is correct, we call C convex. Meanwhile, C is called homogeneous concave (or convex), if C is both positively homogeneous and concave (or convex).

Theorem 3.1 Consider the parametric optimization problem $P(u)$. If f is homogeneous quasiconvex on $\Omega \times U$, C is homogeneous concave on U, Ω and U are all covex cone, then the optimal function f'' is homogeneous quasiconvex on U.

Proof At first, we will prove that f^{n} is positively homogeneous. For every $a>0$, and $(x,u)\in\Omega\times U$, by the assumptions:

$$f^{*}(au)=\sup_{x\in C(au)}f(x,au)=\sup_{x\in C(u)}f(ax,au)=\sup_{x\in C(u)}f(x,u)=af^{n}(u).$$

Second, f^{n} is also quasiconvex. In fact, for every $a,\beta>0, a+\beta=1$ and or all $u_1, u_2\in U$:

$$\begin{aligned}f^{n}(au_1+\beta u_2)&=\sup_{x\in C(au_1+\beta u_2)}f(x,au_1+\beta u_2)\\&<\sup_{\substack{x_1\in C(u_1)\\x_2\in C(u_2)}}f(ax_1+\beta x_2,\ au_1+\beta u_2)\\&<\max\{\sup_{x_1\in C(u_1)}f(x_1,u_1),\ \sup_{x_2\in C(u_2)}f(x_2,u_2)\}\\&=\max\{f^{n}(u_1),\ f^{n}(u_2)\}\end{aligned}$$

We notice relation of homogeneous concavity and sublinear propety of the set value mapping the following result is now immediate.

Theorem 3.2 Consider problem $P(u)$, If f is jointly homogeneous quasiconvex on $\Omega\times U$, constraint set value mapping is sublinear on U, Ω is convex sex, U is vector subspace in R^{n}, then f^{n} is homogeneous quasiconvex on U.

The slight extension of the notion of a set value mapping is given by next definition.

Definition 3.3 Set value mapping C: $U\subset R^{n}\to P(R^{n})$ is called quasiconvex. If for every $a,\beta>0$ $a+\beta=1$, and for all $u_1, u_2\in U$:

$$C(au_1+\beta u_2)\supset\min[C(u_1)\vee C(u_2)]$$

Also, if the following contain relation is right.

$$C(au_1+\beta u_2)\subset\max[C(u_1)\vee C(u_2)]$$

then we call C quasiconcave, where $\min[C(u_1)\vee C(u_2)]$ and $\max[C(u_1)\vee C(u_2)]$ is respectively on behalf of the minimum and maximum of $C(u_1)$ and $C(u_2)$.

If set value mapping C is both positively homogeneous and quasiconcave (quasiconvex), we call it homogeneous quasiconcave (or quasiconvex).

Theorem 3.3 For the parametric programming problem $P(u)$. If f is jointly homogeneous quasiconvex on set $\{(x,u)\mid x\in C(u)\}$, constraint set value mapping C is homogeneous quasiconcave on U, U is a convex cone, then optimal value function f^{n} is also homogeneous quasiconvex on U.

Proof For all $a,\beta>0$, $a+\beta=1$, and $u_1, u_2\in U$ by the assumptions:

$$\begin{aligned}f^{n}(au_1+\beta u_2)&=\sup_{x\in C(au_1+\beta u_2)}f(x,au_1+\beta u_2)<\sup_{x\in\max[C(u_1)\vee C(u_2)]}f(x,au_1+\beta u_2)\\&<\max\{\sup_{x\in C(u_1)}f(x,u_1),\ \sup_{x\in C(u_2)}f(x,u_2)\}=\max\{f^{n}(u_1),\ f^{n}(u_2)\}\end{aligned}$$

i.e., f^{n} is quasiconvex on U the proof of positively homogeneous is similar to Theorem 3.1. thus completing the proof of the Theorem.

Remark 3.1 It is rather easy to obtain specializations of the above results to homogeneous quasiconcavity (or quasiconvexity) at a point and uniform quasiconvexity. The above result to the parametric optimization problems $P_3(u)$ and $P'(u)$ can be easily obtained in a manner analogous to $P(u)$. But it is rather difficult to generatize $P_1(u)$ and $P_2(u)$. This is because the sublinear property and homogeneous concavity (or quasiconcavity) of constraint set value mapping C can be not given by the same properties of g and h.

Remark 3.2 The above results are not applicable for homogeneous concave (quasiconcave) case, but we add to a fundamental assumption condition e. g., for all $u_1, u_2 \in U$, $x_i \in C(u_1)$ $(i=1,2,)$ the following inequality is correct:

$$f(x_1, u_1) \geqslant (\text{or} \leqslant) f(x_2, u_2)$$

The results to homogeneous quasiconcavity can be stated in a manner analogous to homogeneous quasiconvexity specially.

Remark 3.3 We may state deeply homogeneous quasiconcavity (or quasiconvexity) properties of the optimal function under correspondind the some special structrues of the constraint set value mapping, such as homogeneous quasimonotoic homogeneous topological closure concave (convex), homogeneous hull concave (convex) and homogeneous closure hull quasiconcave (quasiconvex) case, etc..

References

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参数规划中最优值函数的次线性及齐次拟凹凸性

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摘　　要

本文主要对参数最优化问题 $P(u)$：

$$\max f(x, u)$$
$$\text{s.t. } x \in C(u)$$

的最优值函数的次线性和齐次拟凹凸性进行了系统研究，同时还探讨了通过特殊化 $P(u)$ 的目标函数或约束条件而得到的其它几个参数最优化问题．许多新结果对一般的抽象空间，如线性空间、线性拓扑空间、线性赋范空间或Banach空间等亦是有效的.有关结论可应用于许多最优控制问题和经济数学，也可应用到分式规划的研究中去．