Journal of Mathematical Research and Exposition
Vol.10 No.3 Aug. 1990

On the Durrmeyer Operator over Arbitrary Triangles*

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1. Introduefion

Let f be an integrable function on $[0,1]$, then the Durrmeyer operator $M_n(*, f)$ applied to f is

$$M_n(f; x) = (n+1)\sum_{i=0}^{n} P_{ni}(x)\int_0^1 f(t)\,P_{ni}(t)\,\mathrm{d}t,$$

where $P_{ni}(x) = \binom{n}{i}x^i(1-x)^{n-i}$. This operator was first introduced by Durrmeyer [1], then Derriennic [2] and Gao [3] studied the approximate properties of $M_n(*, f)$ in spaces $C[0,1]$, $C^k[0,1]\,(k\geqslant 1)$, $L_p[0,1]$, $BV[0,1]$ and Sobolev space $W_{rp}[0,1]$ respectively In this paper we shall generalize the operator M_n on $[0,1]$ to the triangles in plane.

Let T_1, T_2, T_3 be the vertices of triangle T. We shall identify any point p in T with its barycentric coordinates (u, v, w) and write $p = p(u, v, w)$, where

$$p = uT_1 + vT_2 + wT_3,\quad u+v+w=1,\quad u\geqslant 0,\quad v\geqslant 0,\quad w\geqslant 0.$$

A function f defined on T can be expressed in terms of the barycentric coordinates of p, i.e. $f(p) = f(u, v, w)$, then the Durrmeyer operator $M_n(f; p)$ over triangle T is given by

$$M_n(f(p'), p) = \frac{(n+1)(n+2)}{\sqrt{3}}\sum_{i+j+k=n} J^n_{i,j,k}(p)\iint_T J^n_{i,j,k}(p')\,f(p')\,\mathrm{d}p'. \tag{1}$$

Where $J^n_{i,j,k}(P) = n!/i!\,j!\,k!\,u^i v^j w^k$ is the Bernstein basis polynomials, the integral is surface integral over triangle T.

The purpose of this paper is that we shall discuss a series of approximate properties of the operator (1) in space $C^k(T)$, $L_p(T)$ and $W_{rp}(T)$, etc.

2. Error estimate in space $C(T)$

Suppuse that X is a compact set in a normed linear space. We denote by $C(X)$ the space of all continuous functions on X, the modulus of continuity of f denoted by $\omega_f(\cdot)$, $\mathscr{L}: C(X)\to C(X)$ is a linear posittve operator. Then we have

Lemma 1 Let X be a convex set, then

* Received Sep. 9, 1988. Projects supported by the science fund of the Jiangsu province.

$$\omega_f(\lambda\delta)<(1+[\lambda])\omega_f(\delta),$$

where $[\lambda]$ denotes integet part of λ.

Lemma 2 If $f,g\in C(X)$, then

$$\mathscr{L}^2\{fg;x\}<\mathscr{L}\{f^2;x\}\cdot\mathscr{L}\{g^2;x\}.$$

Lemma 3 If a sequence of the linear positive operators $\{\mathscr{L}_n(f;x)\}$ $(n=1,2,3,\cdots)$ satisfies the conditions on X:

(i) $\qquad \mathscr{L}_n\{1;x\}=1;$

(ii) $\qquad \mathscr{L}_n\{\|t-x\|^2;x\}<\dfrac{A}{\beta_n}, \qquad (A>0)$

whhere X is a convex set. Then for any $f\in C(X)$, we have

$$\|\mathscr{L}_n\{f(t);x\}-f(x)\|_\infty<\min(A,\sqrt{A})\,\omega_f\left(\frac{1}{\sqrt{\beta_n}}\right). \tag{2}$$

Proof Writing

$$\mathop{\mathscr{L}_n}_{\sqrt{\beta_n}\|t-x\|>1}\{f(t);x\}=\begin{cases}\mathscr{L}_n\{f(t);x\}, & \text{if } \sqrt{\beta_n}\|t-x\|>1,\\ 0 & \text{if } \sqrt{\beta_n}\|t-x\|<1.\end{cases}$$

Observe that, $\mathop{\mathscr{L}_n}_{\sqrt{\beta_n}\|t-z\|>1}$ is also a linear positive operator. Then for $A<1$,

$$\begin{aligned}
&\|\mathscr{L}_n\{f(t);x\}-f(x)\|_\infty<\mathscr{L}_n\{\omega_f(\|t-x\|,x)\}\\
&<\{1+\mathscr{L}_n[\sqrt{\beta_n}\|t-x\|;x]\}\omega_f\left(\frac{1}{\sqrt{\beta_n}}\right)\\
&<\{1+\mathop{\mathscr{L}_n}_{\sqrt{\beta_n}\|t-x\|>1}\{\sqrt{\beta_n}\|t-x\|;x\}\}\omega_f\left(\frac{1}{\sqrt{\beta_n}}\right)\\
&<\{1+\mathop{\mathscr{L}_n}_{\sqrt{\beta_n}\|t-x\|>1}\{\beta_n\|t-x\|^2;x\}\}\omega_f\left(\frac{1}{\sqrt{\beta_n}}\right)<(1+A)\,\omega_f\left(\frac{1}{\sqrt{\beta_n}}\right),
\end{aligned} \tag{3}$$

for $A>1$, using the lemma 2, we have

$$\begin{aligned}
&\|\mathscr{L}_n\{f(t);x\}-f(x)\|_\infty<\{1+\mathscr{L}_n\{\sqrt{\beta_n}\|t-x\|;x\}\}\omega_f\left(\frac{1}{\sqrt{\beta_n}}\right)\\
&<\{1+[\mathscr{L}_n\{\beta_n;x\}]^{\frac{1}{2}}\cdot[\mathscr{L}_n\{\|t-x\|^2;x\}]^{\frac{1}{2}}\}\,\omega_f\left(\frac{1}{\sqrt{\beta_n}}\right)<(1+\sqrt{A})\,\omega_f\left(\frac{1}{\sqrt{\beta_n}}\right).
\end{aligned} \tag{4}$$

By (3) and (4), we get (2).

The following lemmas are easily shown:

Lemma 4 $\forall a,\beta,\gamma>0$, we have

$$\iint_T u^a v^\beta w^\gamma \,\mathrm{d}p=\sqrt{3}\,\frac{\Gamma(1+a)\Gamma(1+\beta)\Gamma(1+\gamma)}{\Gamma(3+a+\beta+\gamma)}$$

Lemma 5 The following equalities are true:

(i) $\qquad M_n(1;x)=1,$

(ii) $\qquad M_n(\|p'-p\|^2;p)=\dfrac{2(1-u^2-v^2-w^2)n+12(u^2+v^2+w^2)-2}{(n\ \ 3)(n+4)}$

By lemmas (3) and (5) above, we directly obtain the following theorem:

Theorem 1 If $f(p)=f(u,v,\omega)\in C(T)$, then for $n>1$

$$\|M_n(f(p');p)-f(p)\|_\infty < (1+\frac{2\sqrt{3}}{3})\omega_f(\frac{1}{\sqrt{n+4}}) \tag{5}$$

3. Convergence of derivative of $M_n(f;p)$

Now we discuss the convergent property of the derivative of $M_n(f;p)$. In this section, we assume that

$$f(u,v,w)=f(u,v,1-u-v).$$

Lemma 6 $\forall r,s\in N$, $r+s<n$, the following equality holds:

$$\frac{\partial^{r+s}}{\partial u^r\partial v^s}M_n(f(p');p)=\frac{n!(n+2)!}{\sqrt{3}(n-r-s)!(n+r+s)!}(-1)^{r+s}\cdot$$

$$\sum_{i+j+k=n-r-s}J_{i,j,k}^{n-r-s}(p)\iint_T\{\frac{\partial^{r+s}}{\partial u'^r\partial v'^s}J_{i+r,j+s,k+r+s}^{n+r+s}(p')\}f(p')\,dp' \tag{6}$$

Proof Using the Leibniz formula and inverting the order of summations, we get

$$\frac{\partial^r}{\partial u^r}M_n(f(p');p)$$

$$=\frac{(n+2)!}{\sqrt{3}(n-r)!}\sum_{0<i+k<n-r}(-1)^rJ_{i,j-r,k}^{n-r}(p)\iint_T\{\sum_{l=0}^{r}(-1)^l\binom{r}{l}J_{i+l,j-r,k+r-l}^{n}(p')\}f(p')\,dp'$$

$$=(-1)^r\cdot\frac{n!(n+2)!}{\sqrt{3}(n-r)!(n+r)!}\sum_{0<i+k<n-r}J_{i,j-r,k}^{n-r}(p)F_{i,j,k}^{*},$$

where $$F_{i,j,k}^{*}=\iint_T\{\frac{\partial^r}{\partial u^r}J_{i,j-r,k}^{n-r}(p')\}f(p')\,dp'.$$

Using the Leibniz formula again and inverting the order of summations four times, we may acquire

$$\frac{\partial^{r+s}}{\partial u^r\partial v^s}M_n\{f(p');p\}$$

$$=\frac{n!(n+2)!}{\sqrt{3}(n-r-s)!(n+r)!}\sum_{m=0}^{s}\sum_{k=s-m}^{n-r-m}\sum_{i=0}^{n-r-m-k}(-1)^{r+s-m}\binom{s}{m}J_{i,j-r-m,k-s+m}^{n-r-s}(p)F_{i,j,k}^{*}$$

$$=(-1)^{r+s}\frac{n!(n+2)!}{\sqrt{3}(n-r-s)!(n+r)!}\sum_{0<i+k<n-r-s}J_{i,j-r-s,k}^{n-r-s}(p)\iint_T\frac{\partial^r}{\partial u^r}\{\sum_{m=0}^{s}(-1)^m\binom{s}{m}\cdot$$

$$\cdot J_{i+r,j-r-s+m,k+r+s-m}^{n+r}(p')\}f(p')\,dp'. \tag{7}$$

But easily note that,

$$\frac{\partial^s}{\partial v^s}J_{i+r,j-r,k+r+s}^{n+r+s}(p)=\frac{(n+r+s)!}{(n+r)!}\sum_{m=0}^{s}(-1)^m\binom{s}{m}J_{i+r,j-r-s+m,k+r+s-m}^{n+r}. \tag{8}$$

Hence (6) is a consequence of (7) and (8).

Now we consider the operator

$$M_n^*(f;p)=\frac{(n+r+s+1)(n+r+s+2)}{\sqrt{3}}\sum_{i+j+k=n-r-s}J_{i,j,k}^{n-r-s}(p)\iint_T J_{i+r,j+s,k+r+s}^{n+r+s}(p')f(p')\,dp', \tag{9}$$

where $r,s\in N$ and $0\leqslant r+s<n$. It will be observad that $M_n^*(f;p)$ are positive linear operators too. Using lemma 3 again, we may prove the following theorem:

Theorem 2 Let r,s be any fixed positive integer such that $r+s<n$; $f\in C(T)$, then for all n large enough,

$$\|M_n^*(f;p)-f(p)\|_\infty\leqslant(1+\sqrt{2})\,\omega_f\left(\frac{1}{\sqrt{n}}\right). \qquad (10)$$

From this, we get

Theorem 3 If $\forall r',s'\in N$, $r'<r$, $s'<s$,

$$\frac{\partial^{r'+s'}}{\partial u^{r'}\partial v^{s'}}f(p)\in C(T),$$

then for all n large enough, we have

$$\left\|\frac{(n-r-s)!\,(n+r+s+2)!}{n!\,(n+2)!}\frac{\partial^{r+s}}{\partial u^r\partial v^s}M_n(f;p)-\frac{\partial^{r+s}}{\partial u^r\partial v^s}f(p)\right\|_\infty\leqslant(1+\sqrt{2})\,\omega^*\left(\frac{1}{\sqrt{n}}\right). \qquad (11)$$

Where $\omega^*(\cdot)$ are the modulus of continuity of $\frac{\partial^{r+s}}{\partial u^r\partial v^s}f(p)$.

Corollarg 1 If for any $r'<r$, $s'<s$, $\frac{\partial^{r'+s'}}{\partial u^{r'}\partial v^{s'}}f(p)\in C(T)$, then $\frac{\partial^{r+s}}{\partial u^r\partial v^s}M_n(f;p)\to\frac{\partial^{r+s}}{\partial u^r\partial v^s}f(p)$ $(n\to\infty)$ uniformly on T.

3. Asymptotic formula

By means of the methods similar to (6) or (7), we get

Theorem 4 If all second order partial derivatives of f are continuous on T, then $\forall p\in T$ holds

$$\begin{aligned}M_n(f(p');p)=&f(p)+\frac{1}{n}\left[(1-3u)\frac{\partial f}{\partial u}+(1-3v)\frac{\partial f}{\partial v}+(1-3w)\frac{\partial f}{\partial w}\right]\\&+\frac{1}{n}\left[u(1-u)\frac{\partial^2 f}{\partial u^2}+v(1-v)\frac{\partial^2 f}{\partial v^2}+w(1-w)\frac{\partial^2 f}{\partial w^2}\right.\\&\left.-2uv\frac{\partial^2 f}{\partial u\partial v}-2vw\frac{\partial^2 f}{\partial v\partial w}-2wu\frac{\partial^2 f}{\partial w\partial u}\right]+o\left(\frac{1}{n}\right)\end{aligned} \qquad (12)$$

4. The convergence in space $L_p(T)$ and $W_{r,p}(T)$.

First of all, let's discuss the convergence in space $L_p(T)$. The norm of $f\in L_p(T)$ is denoted by $\|f\|_{L_p(T)}$, number q is conjugate exponent to p.

Lemma 7 $\forall f\in L_p(T)$ and $1\leqslant p\leqslant+\infty$ we have

$$\|M_n(f(Q');Q)\|_{L_p(T)}\leqslant\|f\|_{L_p(T)}. \qquad (13)$$

Proof If $p=+\infty$, since $\|f\|_{L_\infty(T)}=\operatorname{ess\,sup} f(Q)$, the inequality holds certainly. If $p=1$ since

$$\|M_n(f(Q');Q)\|_{L_1(p)}$$
$$=\iint_T\left|\frac{(n+1)(n+2)}{\sqrt{3}}\sum_{i+j+k=n}J^n_{i,j,k}(Q)\iint_T J^n_{i,j,k}(Q')f(Q')dQ'\right|dQ$$

$$< \frac{(n+1)(n+2)}{\sqrt{3}} \iint_T \{ \sum_{i+j+k=n} J^n_{i,j,k}(Q) \iint_T J^n_{i,j,k}(Q') |f(Q')| dQ' \} dQ$$
$$= \iint_T |f(Q')| dQ' = \| f \|_{L_1(T)},$$

the inequality (13) holds too. Now let $1 < p < +\infty$, the Hölder inequality yields

$$\{ \sum_{i+j+k=n} J^n_{i,j,k}(Q) \iint_T J^n_{i,j,k}(Q') | f(Q') | dQ' \}^p$$
$$< \sum_{i+j+k=n} J^n_{i,j,k}(Q) (\iint_T J^n_{i,j,k} Q') | f(Q') | dQ')^p$$
$$< \left(\frac{\sqrt{3}}{(n+1)(n+2)} \right)^{p-1} \sum_{i+j+k=n} J^n_{i,j,k}(Q) \iint_T J^n_{i,j,k}(Q') |f(Q')|^p dQ'.$$

From this we get

$$\| M_n(f(Q'); Q) \|_{L_p(T)}$$
$$< \frac{(n+1)(n+2)}{\sqrt{3}} \{ \iint_T \left(\frac{\sqrt{3}}{(n+1)(n+2)} \right)^{p-1} \sum_{i+j+k=n} J^n_{i,j,k}(Q) \iint_T J^n_{i,j,k}(Q') |f(Q')|^p dQ' \}^{\frac{1}{p}}$$
$$= \| f \|_{L_p(T)}.$$

The proof of Lemma 7 is complete.

Theorem 5 Let $f \in L_p(T)$, then

$$\lim_{n \to \infty} \| M_n(f(Q'); Q) - f(Q) \|_{L_p(T)} = 0. \tag{14}$$

Proof Since $f \in L_p(T)$, then $\forall \varepsilon > 0$ there exists a continuous function g with a compact support in T, such that

$$\| f - g \|_{L_p(T)} < \varepsilon.$$

Since $g \in C(T)$, then by the theorem 1, $\exists N$, such that $\forall n > N$

$$\| M_n(g; Q) - g(Q) \|_\infty < \varepsilon.$$

Now by the Minkowski inequality and lemma 7,

$$\| M_n(f; Q) - f(Q) \|_{L_p(T)}$$
$$< \| M_n(f; Q) - M_n(g\ Q) \|_{L_p(T)} + \| f - g \|_{L_p(T)} + \| M_n(g; Q) - g(Q) \|_{L_p(T)}$$
$$< 2 \| f - g \|_{L_p(T)} + \| M_n(g; Q) - g(Q) \|_{L_p(T)} < (2 + |T|^{\frac{1}{p}}) \varepsilon$$

where $|T|$ is the area of triangle T. This proves (14).

Finally, let us deal with the convergence in Sobolev space $W_{pr}(T)$. Let r_1, r_2 are nonnegative integers, $r = (r_1, r_2)$, $|r| = r_1 + r_2$, $D^r f = (\partial^{|r|} / \partial u^r \partial v^r) f$, the norm of $f(p) \in W_{pr}(T)$ is denoted by $\| f \|_{W_{pr}(T)}$,

$$\| f \|_{W_{pr}(T)} = \left(\sum_{|m| < |r|} \| D^m f \|^p_{L_p(T)} \right)^{\frac{1}{p}}.$$

In this case, we obtain results similar to in space $L_p(T)$ as follows:

Lemma 8 If $f(Q)\in W_{pr}(T)$, then

$$\| M_n(f(Q');Q)\|_{W_{pr}(T)} < \| f\|_{W_{pr}(T)} .$$

Theorem 6 If $f(Q)\in W_{pr}(T)$, then we have

$$\lim_{n\to\infty}\| M_n f;Q) - f(Q)\|_{W_{pr}(T)} = 0.$$

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关于任意三角形区域上的Durrmeyer算子

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摘　　要

本文将区间〔0,1〕上的Durrmeyer算子推广到平面上的任意三角形区域中去（见正文（1）式），并在空间$C(T)$,$C^k(T)$（$k\geq 1$），$L_p(T)$以及Sobolev空间 $W_{pr}(T)$中研究了它的收敛性及逼近度估计，这里T是平面上的三角形区域.