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# A Note on Symmetric Block Circulant Matrix*

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The symmetrc block circulant matix is a useful tool in the vibration analysis of structures. For example, the computation of the natural frequencies and the corresponding normal modes for rotationally periodic structures can simply be reduced to the solution of a generalized eigenvalue problem with a pair of symmetric block circulant matrices. Other applications of circulant matrix see [1,3].

Chao [2] has studied the spectrum properties of symmetric circulant matrix. But his results on the symmetric block matrix with ciruulant blocks are incorrect. In this note we study the eigenvalue problem for symmetric block circulant matrix. When each block in it is a circulant matrix, our result corrected the errors in [2].

Let $A\in\mathscr{B}\mathscr{C}(m, n)$ be a block circulant matrix:

$$A=\begin{bmatrix} A_0 & A_1 & \cdots & A_{m-1} \\ A_{m-1} & A_0 & \cdots & A_{m-2} \\ \vdots & \vdots & \vdots & \vdots \\ A_1 & A_2 & \cdots & A_0 \end{bmatrix}$$

where $A_k\in R^{n\times n}$, $k=0, \cdots, m-1$.

Obviously, $A$ can be expressed as follows:

$$A=\sum_{k=0}^{m-1}\Pi_m^k\otimes A_k, \tag{1}$$

where $\Pi_m\in R^{m\times m}$ is a permutation matrix:

$$\Pi_m=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2}$$

and $\otimes$ denotes Kronecker product.

**Lemma 1** $A\in\mathscr{B}\mathscr{C}(m, n)$ is a symmetric matrix if and only if

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$$A_k = A_{m-k}^T, \quad k = 1, \cdots, m-1, \tag{3}$$

and

$$A_0 = A_0^T. \tag{3}$$

**Proof** From ( 1 ) and the well known properties of Kronecker product we have

$$A^T = \left(\sum_{k=0}^{m-1} \Pi_m^k \otimes A_k\right)^T = \sum_{k=0}^{m-1} (\Pi_m^k)^T \otimes A_k^T = \sum_{k=0}^{m-1} \Pi_m^{m-k} \otimes A_k^T.$$

If conditions ( 3 ) are satisfied we get immediately:

$$A^T = \sum_{k=0}^{m-1} \Pi_m^{m-k} \otimes A_{m-k} = A. \quad (A_m \equiv A_0).$$

Conversely, if $A^T = A$, then we have

$$\sum_{k=0}^{m-1} \Pi_m^{m-k} \otimes A_k^T = \sum_{k=0}^{m-1} \Pi_m^k \otimes A_k,$$

thus, from ( 2 ) we can deduce ( 3 ).

Let $F_l \in C^{l \times l}$ be a matrix as follows:

$$F_l = \frac{1}{\sqrt{l}}(\omega_l^{(r-1)(s-1)}) = \frac{1}{\sqrt{l}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_l & \omega_l^2 & \cdots & \omega_l^{l-1} \\ 1 & \omega_l^2 & \omega_l^4 & \cdots & \omega_l^{2(l-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \omega_l^{l-1} & \omega^{2(l-1)} & \cdots & \omega^{(l-1)^2} \end{bmatrix} \tag{4}$$

where $\omega_l = \exp\left(\frac{2\pi i}{l}\right)$. Obviously, $F_l$ is a unitary matrix.

**Theorem 2** Let $A \in \mathscr{BC}(m, n)$ be symmetric, then $A$ is unitarily similar to an Hermitian block diagonal matrix, i.e., $A$ is of the form:

$$A = (F_m \otimes I_n)\, \mathrm{diag}(\widetilde{M}_0, \cdots, \widetilde{M}_{m-1})(F_m \otimes I_n)^*, \tag{5}$$

where $\widetilde{M}_j \in C^{n \times n}$, $j = 0, \cdots, m-1$, are as follows:

( 1 ) For $m$ even$\geqslant 2$, $\widetilde{M}_j$ is the following matrix:

$$\widetilde{M}_j = A_0 + \sum_{k=1}^{m/2-1} (\omega_m^{kj} A_k + \bar{\omega}_m^{kj} A_k^T) + (-1)^j A_{m/2}. \tag{6}$$

( 2 ) For $m$ odd$\geqslant 3$, $\widetilde{M}_j$ is the following matrix:

$$\widetilde{M}_j = A_0 + \sum_{k=1}^{(m-1)/2} (\omega_m^{kj} A_k + \bar{\omega}_m^{kj} A_k^T). \tag{7}$$

**Proof** Let $\Omega_m$ be the following matrix:

$$\Omega_m = \mathrm{diag}(1, \omega_m, \omega_m^2, \cdots, \omega_m^{m-1}), \tag{8}$$

then we have

$$\Pi_m = F_m \Omega_m F_m^*.$$

Hence, it holds

$$\Pi_m^k \otimes A_k = (F_m \Omega_m^k F_m^*) \otimes F_n (F_n^* A_k F_n) F_n^* .$$

From ( 1 ) we have

$$A = (F_m \otimes F_n) \left[ \sum_{k=0}^{m-1} \Omega_m^k \otimes (F_n^* A_k F_n) \right] (F_m \otimes F_n)^* . \tag{9}$$

For $m$ even and $>2$. By Lemma 1 we have

$$\begin{aligned} \sum_{k=0}^{m-1} \Omega_m^k \otimes (F_n^* A_k F_n) &= I_m \otimes (F_n^* A_0 F_n) \\ &+ \sum_{k=1}^{m/2-1} [\Omega_m^k \otimes (F_n^* A_k F_n) + \overline{\Omega}_m^k \otimes (F_n^* A_k F_n)^*] \\ &+ \Omega_m^{m/2} \otimes (F_n^* A_{m/2} F_n) = \operatorname{diag}(M_0, M_1, \cdots, M_{m-1}) . \end{aligned} \tag{10}$$

where, for $j = 0, 1, \cdots, m-1$,

$$\begin{aligned} M_j &= F_n^* A_0 F_n + \sum_{k=1}^{m/2-1} [\omega_m^{kj} (F_n^* A_k F_n) + \overline{\omega}_m^{kj} (F_n^* A_k F_n)^*] + (-1)^j F_n^* A_{m/2} F_n \\ &= F_n^* [A_0 + \sum_{k=1}^{m/2-1} (\omega_m^{kj} A_k + \overline{\omega}_m^{kj} A_k^T) + (-1)^j A_{m/2}] F_n = F_n^* \widetilde{M}_j F_n . \end{aligned} \tag{11}$$

Thus, we have

$$\operatorname{diag}(M_0, M_1, \cdots, M_{m-1}) = (I_m \otimes F_n^*) \operatorname{diag}(\widetilde{M}_0, \cdots, \widetilde{M}_{m-1}) (I_m \otimes F_n) .$$

From ( 9 ), (10) and (11) we get ( 5 ) with ( 6 ).

Analogously, for $m$ odd $>3$, we have

$$\begin{aligned} &\sum_{k=0}^{m-1} [\Omega_m^k \otimes (F_n^* A_k F_n)] = I_m \otimes (F_n^* A_0 F_n) \\ &+ \sum_{k=1}^{(m-1)/2} [\Omega_m^k \otimes (F_n^* A_k F_n) + \overline{\Omega}_m^k \otimes (F_n^* A_k F_n)^*] = \operatorname{diag}(M_0, \cdots, M_{m-1}), \end{aligned} \tag{12}$$

where, for $j = 0, \cdots, m-1$,

$$M_j = F_n^* [A_0 + \sum_{k=1}^{(m-1)/2} (\omega_m^{kj} A_k + \overline{\omega}_m^{kj} A_k^T)] F_n = F_n^* \widetilde{M}_j F_n . \tag{13}$$

From ( 9 ), (12) and (13) we get ( 5 ) with ( 7 ).

Let $B \in \mathscr{BC}(m, n)$ be a block circulant matrix:

$$B = \begin{bmatrix} B_0 & B_1 & \cdots & B_{m-1} \\ B_{m-1} & B_0 & \cdots & B_{m-2} \\ \cdots & \cdots & \cdots & \cdots \\ B_1 & B_2 & \cdots & B_0 \end{bmatrix}$$

where $B_k \in R^{n \times n}$, $k = 0, \cdots, m-1$.

**Theorem 3** Let $A, B \in \mathscr{BC}(m, n)$ be symmetric, then the symmetric matrix pencil $A - \lambda B$ is unitarily equivalent to the following symmetric matrix pencil

$$\operatorname{diag}(\widetilde{M}_0 - \lambda \widetilde{N}_0, \cdots, \widetilde{M}_{m-1} - \lambda \widetilde{N}_{m-1}),$$

where $\widetilde{M}_j$, $j=0,1,\cdots,m-1$, are the same as in (6) or (7), while $\widetilde{N}_j$, $j=0,1,\cdots,m-1$, are analogously defined for $B$ as $\widetilde{M}_j$ for $A$.

Furthermore, if $A$ (or $B$) is a symmetric positive definite matrix (in such case $A-\lambda B$ is called symmetric positive definite pencil), then each pencil

$$\widetilde{M}_j-\lambda\widetilde{N}_j \quad (j=0,\cdots,m-1)$$

is a symmetric positive definite one.

**Proof** Apply Theorem 2 for $A$ and $B$.

**Theorem 4** Let $A\in\mathscr{B}\mathscr{C}(m,n)(n\geqslant 2)$ be symmetric with each $A_k\in R^{n\times n}$ a circulant matrix for $k=0,1,\cdots,m-1$. Then for the $mn$ eigenvalues $\lambda_{j,s}$, $j=0,1,\cdots,m-1$, $s=0,1,\cdots,n-1$, of $A$ we have:

(1) For the case of $m$ an even integer $\geqslant 2$:

(a) For $n$ even and $\geqslant 2$:

$$\lambda_{j,s}=a_0^{(0)}+(-1)^s a_{n/2}^{(0)}+2\sum_{r=1}^{n/2-1}a_r^{(0)}\cos\frac{2sr\pi}{n}+2\sum_{k=1}^{m/2-1}\sum_{r=0}^{n-1}a_r^{(k)}\cos\left(2\left(\frac{kj}{m}+\frac{rs}{n}\right)\pi\right)$$

$$+(-1)^j\left[a_0^{(m/2)}+(-1)^s a_{n/2}^{(m/2)}+2\sum_{r=1}^{n/2-1}a_r^{(m/2)}\cos\frac{2sr\pi}{n}\right].$$

(b) For $n$ odd and $\geqslant 3$:

$$\lambda_{j,s}=a_0^{(0)}+2\sum_{r=1}^{(n-1)/2}a_r^{(0)}\cos\frac{2sr\pi}{n}+2\sum_{k=1}^{m/2-1}\sum_{r=0}^{n-1}a_r^{(k)}\cos\left(2\left(\frac{kj}{m}+\frac{rs}{m}\right)\pi\right)$$

$$+(-1)^j\left[a_0^{(m/2)}+2\sum_{r=1}^{(n-1)/2}a_r^{(m/2)}\cos\frac{2sr\pi}{n}\right].$$

(2) For the case of $m$ an odd integer $\geqslant 3$:

(a) For $n$ even and $\geqslant 2$:

$$\lambda_{j,s}=a_0^{(0)}+(-1)^s a_{n/2}^{(0)}+2\sum_{r=1}^{n/2-1}a_r^{(0)}\cos\frac{2sr\pi}{n}+2\sum_{n=1}^{(m-1)/2}\sum_{r=0}^{n-1}a_r^{(k)}\cos\left(2\left(\frac{kj}{m}+\frac{rs}{n}\right)\pi\right).$$

(b) For $n$ odd and $\geqslant 3$:

$$\lambda_{j,s}=a_0^{(0)}+2\sum_{r=1}^{(n-1)/2}a_r^{(0)}\cos\frac{2sr\pi}{n}+2\sum_{k=1}^{(m-1)/2}\sum_{r=0}^{n-1}a_r^{(k)}\cos\left(2\left(\frac{kj}{m}+\frac{rs}{n}\right)\pi\right),$$

where the circulant matrices $A_k\in R^{n\times n}$, $k=0,1,\cdots,m-1$, have the form:

$$A_k=\begin{bmatrix} a_0^{(k)} & a_1^{(k)} & \cdots & a_{n-1}^{(k)} \\ a_{n-1}^{(k)} & a_0^{(k)} & \cdots & a_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & \vdots \\ a_1^{(k)} & a_2^{(k)} & \cdots & a_0^{(k)} \end{bmatrix}$$

**Proof** By Theorem 2 we have:

For $m$ even $\geqslant 2$, the $n$ eigenvalues $\lambda_{j,s}$, $s=0,1,\cdots,n-1$, of $\widetilde{M}_j$ can be expressed as follows (c.f. (6)):

$$\lambda_{j,s} = \lambda_s^{(0)} + 2\sum_{k=1}^{m/2-1} \mathrm{Re}(\omega_m^{kj}\lambda_s^{(k)}) + (-1)^j\lambda_s^{(m/2)}. \tag{14}$$

For $n$ odd $\geqslant 3$, the $n$ eigenvalues $\lambda_{j,s}$, $s=0,1,\cdots,n-1$, of $\widetilde{M}_j$ can be expressed as follows (c.f. (7)):

$$\lambda_{j,s} = \lambda_s^{(0)} + 2\sum_{k=1}^{(m-1)/2} \mathrm{Re}(\omega_m^{kj}\lambda_s^{(k)}), \tag{15}$$

where $\lambda_s^{(k)}$, $s=0,1,\cdots,n-1$, denote the eigenvalues of the circulant matrix $A_k$.

We know from Lemma 1 that $A_0$ and $A_{m/2}$ (when $m$ is even and $\geqslant 2$) are symmetric circulant matrices. From Theorem 2 (Let $m$ be replaced by $n$ and let $n$ equal 1) it is easy to get the following results:

For $n$ even $\geqslant 2$, the $n$ eigenvalues $\lambda_s^{(p)}$, $s=0,1,\cdots,n-1$, of $A_p$, $p=0, m/2$ ($m$ even and $\geqslant 2$) can be expressed as follows:

$$\lambda_s^{(p)} = a_0^{(p)} + (-1)^s a_{n/2}^{(p)} + 2\sum_{r=1}^{n/2-1} a_r^{(p)} \cos\frac{2sr\pi}{n} \tag{16}$$

For $n$ odd $\geqslant 3$, the eigenvalues $\lambda_s^{(p)}$, $s=0,1,\cdots,n-1$, of $A_p$, $p=0, m/2$ ($m$ even and $\geqslant 2$) can be expressed as follows:

$$\lambda_s^{(p)} = a_0^{(p)} + 2\sum_{r=1}^{(n-1)/2} a_r^{(p)} \cos\frac{2sr\pi}{n}. \tag{17}$$

From (14), (15), (16) and (17) the conclusion of the theorem follows.

**References**


[1] R. Bellman, Introduction to matrix analysis, 2nd ed., McGraw-Hill, New York, 1970.

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# 关于对称块轮换矩阵的注记


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**摘　要**

本文研究了对称块轮换矩阵和对称块轮换矩阵束的特征值和广义特征值问题．导出了它们的特征分解．当对称块轮换矩阵的每个块本身也是轮换矩阵时，本文的结果校正了[2]中的错误．