Journal of Mathematical Research and Expostion
Vol.10, No.4, Nov. 1990

The Quasi-Duhamel Principle and Its Applications*

L. C. Hsu (Xu Lizhi) (Dalian Universtiy of Technology)

Sun Guangrun (Qufu Normal University, Qufu)

Abstract

In this paper the techniques of nonstandard analysis are used to establish the quasi-Duhamel Principle. The applications of our result lead to simple nonstandard proofs for some theorems of classical mathematical analysis. This principle is also very useful for nonstandard asymptotic approximation of sums and integrals.

§ 0. Introduction

It is well known [4] that Duhamel's Theorem is a fundamental lemma in the applications of the classical integral calculus. In recent papers [1,2] we generalized it to hyperreal field (which we call here H-field) $^*R$, thus giving the extended Duhamel Principle. The aim of the present paper is to show a more general form of this principle, called the quasi-Duhamel Principle, in which the conditions concerning infinitesimals are left out in particular. It is evident that this new form will have still wider uses, such as to verify directly an important condition which guarantees the existence of Riemann-Stieltjes integral, and other classical results (cf. § 2). Also, it will be expected that this result is a very convenient tool for such problems as counting arguments and that standard infinite structures are approximated by $*$-finite structures.

§ 1. Quasi-Duhamel Principle

Theorem (Quasi-Duhamel Principle). Let $\{s_j\}\ j\in{}^*N$, $\{s_j'\}\ j\in{}^*N$, $\{r_j\}\ j\in{}^*N$ and $\{r_j'\}\ j\in{}^*N$ be four internal sequences in H-field $^*R$ such that for each $j<n$, where n is any finite or infinite natural number,

(i) $s_j'/s_j\simeq 1$,

(ii) $\sum_{j=1}^{n}|s_j|=L$ is finite,

(iii) $r_j\simeq r_j'$,

(iv) r_j and r_j' are finite.

* Received July 2, 1989.

Then the following near-equality holds:

$$\sum_{j=1}^{n} r_j s_j \simeq \sum_{j=1}^{n} r'_j s'_j \qquad (1)$$

and

$$^\circ\left(\sum_{j=1}^{n} r_j s_j\right) = {}^\circ\left(\sum_{j=1}^{n} r'_j s'_j\right) . \qquad (2)$$

In particular, for $n \in {}^*\boldsymbol{N} \setminus \boldsymbol{N}$, (1) and (2) involving two $*$-finite sums are true.

Proof We may consider only the case that n is finite. Since $r'_j \simeq r_j$ and $s'_j/s_j \simeq 1$, $\theta_j = r'_j - r_j$ and $\eta_j = s'_j/s_j - 1$ are infinitesimals. We have $s'_j - s_j = \eta_j s_j$ and so $s'_j = (1+\eta_j)s_j$. Thus

$$r'_j s'_j - r_j s_j = (r'_j - r_j) s'_j + r_j (s'_j - s_j) = \theta_j (1+\eta_j) s_j + r_j \eta_j s_j \qquad (3)$$

By transfer principle, every $*$-finite set of hyperreal numbers has a largest element, therefore we may write

$$\|\theta\| = \max_{1 \le j \le n} \{|\theta_j|\}, \quad \|\eta\| = \max_{1 \le j \le n} \{|\eta_j|\}, \quad \|r\eta\| = \max_{1 \le j \le n} \{|r_j \eta_j|\},$$

which are obviously infinitesimals. Hence we have

$$\left|\sum_{j=1}^{n} r'_j s'_j - \sum_{j=1}^{n} r_j s_j\right| < [\|\theta\|(1+\|\eta\|) + \|r\eta\|] \sum_{j=1}^{n} |s_j| = \varepsilon L \simeq 0,$$

where ε is some infinitesimal. This justifies (1).

It is easily seen the $*$-finite sum $\sum_{j=1}^{n} r_j s_j$ is finite. In fact, by transfer, $\max_{1 \le j \le n} \{|r_j|\} = r$ exists and is finite since every r_j is finite. Thus

$$\left|\sum_{j=1}^{n} r_j s_j\right| < \sum_{j=1}^{\infty} |r_j| |s_j| < rL.$$

Therefore $\sum_{j=1}^{n} r_j s_j$ has standard part. From this fact we obtain (2). ■

§ 2. Application

As applications of the principle we show a few results of the asymptotic approximations and give simple proofs of some known facts about the Riemann-Stieltjes integral.

Ⅰ. Asymptotic approximations of sums

Corollary Ⅰ. In the theorem, if the conditions which r_j and r'_j are finite for all $j \leqslant n$ are left out, then

$$\sum_{j=1}^{n} r'_j s'_j = \sum_{j=1}^{n} (1+\eta_j) r_j s_j , \qquad (4)$$

where each η_j is infinitesimal.

Proof From (3), we have

$$r'_j s'_j - (1+\eta_j) r_j s_j = \theta_j (1+\eta_j) s_j .$$

Therefore (4) is immediate. ■

Corollary 2. (The corollary of dominated approximation) If the conditions (i)—(iv) in the principle are satisfied only for finite natural number n and assume $\sum_{j=1}^{\infty} t_j$ is a standard convergent series such that $|r_j s_j|$, $|r_j' s_j'| < t_j$ for all $j \in {}^*\boldsymbol{N}$, then

$$\sum_{j=1}^{\infty} r_j s_j \simeq \sum_{j=1}^{\infty} r_j' s_j' . \tag{5}$$

Proof **By the quasi**-Duhamel Principle we have

$$\sum_{j=1}^{n} r_j s_j \simeq \sum_{j=1}^{n} r_j' s_j' .$$

By Robinson's Sequential Lemma, there is an infinite natural number w such that

$$\sum_{j=1}^{w} r_j s_j \simeq \sum_{j=1}^{w} r_j' s_j' .$$

Since dominated series $\sum_{j=1}^{\infty} t_j$ converges, $\sum_{j=1}^{\infty} r_j s_j$ and $\sum_{j=1}^{\infty} r_j' s_j'$ converge. Hence $\sum_{j>w}^{\infty} r_j s_j \simeq 0$ and $\sum_{j>\omega}^{\infty} r_j' s_j' \simeq 0$. Thus

$$\sum_{j=1}^{w} r_j s_j \simeq \sum_{j=1}^{\infty} r_j s_j , \quad \sum_{j=1}^{w} r_j' s_j' \simeq \sum_{j=1}^{\infty} r_j' s_j' .$$

It follows from these facts that

$$\sum_{j=1}^{\infty} r_j s_j \simeq \sum_{j=1}^{\infty} r_j' s_j' . \qquad \blacksquare$$

2. Riemann-Stieltjes integral

Theorem. If f is continuous on $[a, b]$ and g is of finite variation (bounded variation) on $[a, b]$, then f is Riemann-integrable with respect to g over $[a,b]$.

Proof **For any** $*$-finite fine partition of $[a, b]$

$$\Delta_w = \{a = t_0 < t_1 < \cdots < t_w = b\} ,$$

such that $t_j \simeq t_{j-1}$ $(1 < j < w)$, where $w \in {}^*\boldsymbol{N} \setminus \boldsymbol{N}$, by the Hyperreal Extreme Value Theorem, the $*$-finite upper Dardoux-Stieltjes sum ${}^*U$ and the $*$-finite lower Darboux-Stieltjes sum ${}^*L$ exist. Thus it follows at once by the quasi-Duhamel Principle that

$${}^*U({}^*f, {}^*g, \Delta_w) \simeq {}^*L({}^*f, {}^*g, \Delta_w) .$$

This shows that $f \in R(g)$ over $[a, b]$. $\blacksquare$

Theorem Assume $f \in R(g)$ over $[a, b]$, and assume that g has a derivative g' on $[a, b]$ and $g' \in R$ over $[a, b]$. Then

$$\int_a^b f(x)\,\mathrm{d}g(x) = \int_a^b f(x) g'(x)\,\mathrm{d}x \tag{6}$$

Proof For any Δ_w as in the preceding theorem, since

$$\Delta_j {}^*g = {}^*g(t_j) - {}^*g(t_{j-1}) = {}^*g'(t_j)\,\Delta t_j + \theta\,\Delta t_j$$

where θ is an infinitesimal and $\Delta t_j = t_j - t_{j-1}$, it is clear that ${}^*g'(t_j)\,\Delta t_j / \Delta_j {}^*g \simeq 1$. Also ${}^*f(\zeta_j) \simeq {}^*f(t_j)$, where $t_{j-1} < \zeta_j < t_j$ for $j < w$, since the function f is uniformly continuous on compact set $[a, b]$. Because g is obviously of finite variation, $\sum_{j=1}^{w} |\Delta_j {}^*g|$ is finite. It is immediately from our principle that

$$\sum_{j=1}^{w} {}^*f(\zeta_j)\,\Delta_j {}^*g \simeq \sum_{j=1}^{w} {}^*f(t_j)\,{}^*g'(t_j)\,\Delta t_j .$$

Taking standard part, (6) is established. ■

Definition (relative derivate). Let F and g be standard real-valued functions on $[a, b] \subseteq \boldsymbol{R}$ and let x_0 be any standard point on $[a, b]$ which belongs to no interval of constancy of the function g. If for every non-zero infinitesimal θ in ${}^*\boldsymbol{R}$, there is a unique real-valued "Monad Function" $f(x_0)$ such that

$${}^\circ\left(\frac{{}^*F(x_0+\theta) - {}^*F(x_0)}{{}^*g(x_0+\theta) - {}^*g(x_0)}\right) = f(x_0),$$

then $F(x)$ is said to be relative differentiable with respect to g at x_0, and $f(x_0)$ is called the relative derivate of F with respect to g at x_0, and write

$$\left.\frac{\mathrm{d}F}{\mathrm{d}g}\right|_{x=x_0} = F'_g(x_0) = f(x_0).$$

If F and g have standard derivete at x_0, it is quite obvious that

$$F'_g(x_0) = \frac{F'(x_0)}{g'(x_0)} \quad (g'(x_0) \neq 0).$$

Definition (relative primitive function). Let F and g be standard real-valued functions on some interval $\Delta \subseteq \boldsymbol{R}$. The function F is called a relative primitive of a function f with respect to g on the interval Δ if f is the relative derivate of F with respect to g on Δ, that is, if $F'_g(x) = f(x)$ for all x on Δ.

Theorem. Assume f is continuous on $[a, b]$ and assume g is of finite variation on $[a, b]$. Let F be any relative primitive of f with respect to g on $[a, b]$. Then

$$\int_a^b f(x)\,\mathrm{d}g(x) = F(b) - F(a).$$

Proof Since $\int_a^b f\,\mathrm{d}g$ exists, we may make such $*$-finite partitton of $[a, b]$ as

$$\Delta_w = \{a = x_0, x_1, x_2, \cdots, x_w = b\},$$
$$x_j = a + (j/w)(b-a),$$
$$j = 0, 1, 2, \cdots, w, \quad w \in {}^*\boldsymbol{N} \setminus \boldsymbol{N},$$

and write $x_j - x_{j-1} = (b-a)/w = \mathrm{d}x$. The difference $\mathrm{d}x$ is obviously infinitesimal.

It follows from the quasi-Duhamel Principle that

$$F(b)-F(a)=\sum_{j=1}^{w}\frac{{}^*F(x_{j-1}+\mathrm{d}x)-{}^*F(x_{j-1})}{{}^*g(x_{j-1}+\mathrm{d}x)-{}^*g(x_{j-1})}\Delta_j{}^*g$$

$$\simeq\sum_{j=1}^{w}{}^*F'_g(x_{j-1})\Delta_j{}^*g=\sum_{j=1}^{w}{}^*f(x_{j-1})\Delta_j{}^*g. \qquad \blacksquare$$

References

[1] Xu Lizhi and Sun Guangrun, The generalized Duhamel Principle and Its Applications (Ⅰ), J. of Qufu Normal Univ. Vol. 14, No.3, 1988, pp. 21—25.

[2] Xu Lizhi and Sun Guangrun, The Generalized Duhamel Principle and Its Applications (Ⅱ), J. of Qufu Normal Univ. Vol.14, No.4, 1988 ,pp. 1—5.

[3] Xu Lizhi, Sun Guangrun and Dong Jiali, Introduction to Modern Infinitesimal Analysis, Dalian University of Technology Press, to appear (1990).

[4] Osgood, W. F., Introduction to the Calculus, The Macmillan Company, New York, 1930, pp. 301—304.

[5] Robinson, A., Nonstandard Analysis, North-Holland, Amsterdam, 1974.

[6] Stroyan K. D., Luxemburg, W. A. J., Introduction to the Theory of Infinitesimals, Academic Press, New York, 1976.

[7] Van der Bery, I., Nonstandard Asymptotic Analysis, Lecture Notes in Mathematics, Vol. 1249, Springer-Verlag, Berlin, 1987, p. 150.

[8] Keisler, H. J., Foundation of Infinitesimal Calculus, Prindle, Weber and Schmidt, Boston, 1976.

[9] Saks, S., Theory of the Integral, Hafner Publishing Company, New York, 1937, p. 272.

[10] Hurd, A. E. and Loeb, P. A., An Introduction to Nonstandard Real Analysis, Academic Press, Inc., 1985.

拟 Duhamel 原理及其应用

徐利治　　孙广润

(大连理工大学)　(曲阜师范大学)

本文应用非标准分析技巧建立了拟Duhamel原理,并利用这一原理简化了经典分析中Riemann-Stieltjes积分的某些定理的证明方法.文中还给出了上述原理对和式渐近逼近的应用.