Journal of Mathematical Research and Exposition
Vol.11 No.1 Feb. 1991

A Spirallikeness Condition for Analytic Functions*

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Let α, β, γ and ρ be real, $|\alpha|<\frac{\pi}{2}$, $\beta>0$, $0\le\rho<1$, and let

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

be analytic in the unit disc $E=\{z:|z|<1\}$. Futher let

$$J(\alpha,\beta,\gamma,\rho,f(z))=\left(e^{i\alpha}\frac{zf'(z)}{f(z)}-\rho\cos\alpha\right)^{1-2\gamma}\Big[\left(e^{i\alpha}\frac{zf'(z)}{f(z)}-\rho\cos\alpha\right)^2$$
$$+\beta\cos\alpha e^{i\alpha}\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\Big]^{\gamma}$$

with $e^{i\alpha}\frac{zf'(z)}{f(z)}-\rho\cos\alpha\neq 0$ and

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)}-\rho\cos\alpha\right)^2+\beta\cos\alpha e^{i\alpha}\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\neq 0$$

in E. Denote by Ω the complex plane slit along the half-lines $\mathrm{Re}w=0$, $\mathrm{Im}w\ge b_1$ and $\mathrm{Re}w=0$, $\mathrm{Im}w\le b_2$, where

$$b_k=x_k\left[\frac{(2(1-\rho)+\beta)x_k^2-2x_k\beta\sin\alpha+\beta((1-\rho)^2\cos^2\alpha+\sin^2\alpha)}{2(1-\rho)x_k^2}\right]^{\gamma}\quad(k=1,2)$$

and where $x_1>0$ and $x_2<0$ are the roots of the equation

$$(2(1-\rho)+\beta)x^2+2x(\gamma-1)\beta\sin\alpha-\beta(2\gamma-1)((1-\rho)^2\cos^2\alpha+\sin^2\alpha)=0.$$

The following result is proved:

Theorem If

$$\mathrm{Re}\,J(\alpha,\beta,\gamma,\rho,f(z))>0,\quad z\in E,\quad \text{for }\gamma\le\frac{1}{2}$$

and

$$J(\alpha,\beta,\gamma,\rho,f(z))\in\Omega,\quad z\in E,\quad \text{for }\gamma>\frac{1}{2},$$

then $f(z)$ is α-spirallike of order ρ, that is,

$$\mathrm{Re}\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\}>\rho\cos\alpha\quad(z\in E).$$

By specializing the parameters involved, several interesting consequences can be deduced[1,2,3].

References

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* Received Apr. 15, 1989.