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The Existence and Nonexistence of a Global Smooth Solution for the Initial Value Problem of Generalized Kuramoto-Sivashinsky Type Equations*

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Abstract

The existence and uniqueness of a global smooth solution for the initial valne problem of generalized Kuramoto-Sivashinsky type equations in multi-dimensions have been obtained. The sufficient conditions of "blowing up" for the solution of some dissipative equations are given.

§ 1 Introduction

The equation

$$\varphi_t + |\nabla_i \varphi|^2 + \Delta\varphi + \Delta^2\varphi = 0 \qquad (1.1)$$

was independently advocated by Kuramoto [1], in connection with reaction-diffusion systems, and by Sivashinsky [2], in modeling flame propagation, it also aries in the context of viscous film flow [3] and bifurcating solutions of the Navier-Stokes equations [4].

Obviously, we can put the K-S equation (1.1) in a conservative form: We differentiate (1.1), obtaining that the new variable vactor $u \equiv u(x,t) = \nabla\varphi = (\varphi_{x_1}, \cdots, \varphi_{x_n})$, satisfies

$$u_t + \nabla|u|^2 + \Delta u + \Delta^2 u = 0 \qquad (1.2)$$

In this paper, we first consider the initial value problem for generalized KS type equation in one dimension

$$u_t + f(u)_x + \alpha u_{xx} + \beta u_{xxxx} = g(u) \qquad (1.3)$$

$$u|_{t=0} = u_0(x), \quad x \in R' \qquad (1.4)$$

where $f(s), g(s)$ are known real functions, α and β are positive constants. The existence and uniqueness of the global smooth solution is one dimension have been obtained in section § 2 and section § 3. In section § 4, we consider the following initial value problem for generalized KS type equations in n dimension

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$$\boldsymbol{u}_t+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\operatorname{grad}\varphi(u_1,\cdots,u_N)+a\Delta\boldsymbol{u}+\beta\Delta^2\boldsymbol{u}=\boldsymbol{g}(\boldsymbol{u}) \quad (1.5)$$

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(x) \quad (1.6)$$

where $\boldsymbol{u}(x,t)=(u_1(x,t),\cdots,u_N(x,t))$ is a N dimensional unknown functional voctor, $\varphi(s_1,\cdots,s_N)$ is a known real function, $\boldsymbol{g}(\boldsymbol{u})=(g_1(u_1,\cdots,u_N),\cdots,g_N(u_1,\cdots,u_N))$ is a N dimensional known functional vector, a and β are constants. Under some conditions on function $\varphi(u_1,\cdots,u_N)$ and functional vector $\boldsymbol{g}(\boldsymbol{u})$, the existence and uniqueness for a global solution for the problem (1.5) (1.6) are obtained. Finally, In section §5, the sufficient conditions of "blowing up" for the solution of some dissipative equations are given.

§ 2 Some Lemmas and a Priori Estimations

In this section, we make a priori estimations for the problem (1.3) (1.4). Let us introduce some functional spaces and notations. Let $C^l(R^n)$ denote the space of functions, l times continuously differentiable over R^n. $L_p(R^n)$ denotes the Lebesque space of measurable functions $u(x,t)$ with p th power abosolute value $|u|$ integrable over R^n with the norm $\|u\|_{L_p(R^n)}=\left(\int_{R^n}|u|^p\mathrm{d}x\right)^{\frac{1}{p}}$. If we define the inner product

$$(u,v)=\int_{R^n}u(x)v(x)\mathrm{d}x,\qquad \|u\|^2_{L_2(R^n)}=(u,u),$$

then $L_2(R^2)$ is a Hilbert space. Let $L_\infty(R^n)$ denotes the Lebesgue space of measurable function $u(x)$ over R^n, which essentially bounded, with the norm

$$\|u\|_{L_\infty}=\operatorname*{ess\,sup}_{x\in R^n}|u(x)|.$$

Let $H^l(R^n)$ denotes the space of the functions with generalized derivatives $D^s u$ $(|s|\le l)$ with the norm

$$\|u\|^2_{H^l(R^n)}=\sum_{|s|\le l}\|D^s u\|^2_{L_2}=\left(\int_{R^n}(1+|\xi|^2)^s|\hat{u}(\xi)|^2\mathrm{d}\xi\right)^{\frac{1}{2}},$$

where $\hat{u}(\xi)$ is the Fourier transformation of $u(x)$. $L^\infty(0,T;H^l(R^n))$ denotes the space of the functions $u(x,t)$ which belong to H^l as a function of x for every fixed t $(0\le t\le T)$ and $\sup\limits_{0\le t\le T}\|u(\cdot,t)\|_{H^l}<\infty$. Especially,

$$\|u\|_{L^\infty(0,T;L_2)}=\sup_{0\le t\le T}\|u\|_{L_2}\quad\text{or}\quad \|u\|_{L_2\times L_\infty}.$$

For the functional vector $\boldsymbol{u}(x,t)=(u_1(x,t),\cdots,u_N(x,t))$

$$\|\boldsymbol{u}(\cdot,t)\|^2_{L_2}=\sum_{i=1}^{N}\|u_i(\cdot,t)\|^2_{L_2}.$$

$\|\boldsymbol{u}(\cdot,t)\|_{H^l}$, $\|\boldsymbol{u}(\cdot,t)\|_{L_\infty}$, and so on are defined similarly.

We always suppose that the function $u(x,t)$ and its derivatives tend to zero as $|x|\to\infty$.

Lemma 1 (Sobolev's inequality [5]) There the constants $\delta>0$, $C>0$ (C depends on δ), such that

$$\|D^k f\|_{L_\infty}\le C\|f\|_{L_2}+\delta\|D^l f\|_{L_2},\quad k<l,\ \forall f\in H^l. \tag{2.1}$$

$$\|D^k f\|_{L_2}\le C\|f\|_{L_2}+\delta\|D^l f\|_{L_2},\quad k\le l,\ \forall f\in H^l. \tag{2.2}$$

Lemma 2 Suppose that $\alpha>0$, $\beta>0$ and assume that (i) $f(u)\in C^1$; (ii) $g(0)=0$, $g'(u)\le b$, $b=\text{const}>0$; (iii) $u_0(x)\in L_2(R^1)$. Then for the solution of problem (1.3)(1.4), there is a solution

$$\begin{aligned}&\|u\|_{L^\infty(0,T;\,L_2(R^1))}\le E_0\\ &\|u_{xx}\|_{L^2(0,T;\,L_2(R^1))}\le E_0\end{aligned} \tag{2.3}$$

where the constant E_0 depends on $\|u_0\|_{L_2(R^1)}$.

Proof Taking the inner product for (1.3) with u, it follows

$$(u_t+f(u)_x+au_{xx}+\beta u_{xxxx},u)=0. \tag{2.4}$$

Since

$$(u,u_t)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L_2}^2,\quad (u,f(u)_x)=-(F(u)_x,1)=0,$$

where

$$F(u)=\int_0^u f(s)\,\mathrm{d}s,\qquad (u,au_{xx})=-a\|u_x\|_{L_2}^2,$$

$$\beta(u,u_{xxxx})=\beta\|u_{xx}\|_{L_2}^2,\quad (u,g(u))\le b\|u\|_{L_2}^2.$$

From Lemma 1 we have

$$\|u_x\|_{L_2}^2\le\varepsilon\|u_{xx}\|_{L_2}^2+c\|u\|_{L_2}^2.$$

Choosing suitably small ε, such that $a\varepsilon\le(1/2)\beta$. Then from (2.4) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L_2}+\frac{\beta}{2}\|u_{xx}\|_{L_2}^2\le(ac+b)\|u\|_{L_2}^2.$$

By using Gronwall inequality, it follows

$$\|u\|_{L_2}^2\le e^{(ac+b)T}\|u_0\|_{L_2}^2=E_0'.$$

$$\begin{aligned}\frac{\beta}{2}\int_0^T\|u_{xx}\|_{L_2}^2\mathrm{d}t&\le(ac+b)\int_0^T\|u\|_{L_2}^2\mathrm{d}t+\|u_0\|_{L_2}^2-\|u\|_{L_2}^2\\ &\le(ac+b)E_0T+2E_0=E_0''.\end{aligned}$$

$$\int_0^T\|u_{xx}\|_{L_2}^2\mathrm{d}t\le\frac{2E_0''}{\beta}.$$

Taking $E_0=\max(E_0',\dfrac{2E_0''}{\beta})$, Lemma 2 follows.

Lemma 3 (Sobolev's estimation [6]) Let $u\in L_q(\Omega)$, $D^m u\in L_r(\Omega)$, $\Omega\le R^n$, $1\le q$, $r\le\infty$. Then there exists a constant C such that

$$\|D^j u\|_{L_p(\Omega)}\le C\|D^m u\|_{L_r(\Omega)}^a\|u\|_{L_q(\Omega)}^{1-a}, \tag{2.5}$$

where $0\le j\le m$, $j/m\le a\le 1$, $1\le p\le\infty$, and

$$\frac{1}{p}=\frac{j}{n}+a\Big(\frac{1}{r}-\frac{m}{n}\Big)+(1-a)\frac{1}{q}.$$

Lemma 4 Suppose that the conditions of lemma 2 are satisfied, and assume that

(1) $f(u)\in C^2$, $|f(u)|\leq A|u|^p$, $1<p<7$,

(2) $u_0(x)\in H^1(R^1)$.

Then for the solution of problem (1.3)(1.4) we have

$$\begin{aligned}&\|u_x\|_{L^\infty(0,T;\,L_2(R^1))}\leq E_1\\&\|u_{xx}\|_{L^2(0,T;\,L_2(R^1))}\leq E_1\end{aligned} \tag{2.6}$$

where the constant E_1 depends on $\|u_0\|_{H^1(R^1)}$.

Proof Differentiating (2.1) with respect to x, and taking the inner product fot the resulting equation with u_x, and letting $v=u_x$, it follows

$$(v_t+f(u)_{xx}+av_{xx}+\beta v_{xxxx},\ v)=(g_u v,v). \tag{2.7}$$

Since

$$(v,v_t)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L_2}^2,\qquad a(v,v_{xx})=-a\|v_x\|_{L_2}^2,$$

$$\beta(v,v_{xxxx})=\beta\|v_{xx}\|_{L_2}^2,\qquad (v,g_u v)\leq b\|v\|_{L_2}^2$$

$$\begin{aligned}(v,f(u)_{xx})&=(v_{xx},f(u))\leq\|v_{xx}\|_{L_2}\|f(u)\|_{L_2}\\&\leq\|v_{xx}\|_{L_2}\Big(\frac{\beta}{6}\|u_{xxx}\|_{L_2}+C\Big)\leq\frac{\beta}{3}\|v_{xx}\|_{L_2}^2+C_1.\end{aligned}$$

Where the following Sobolev's inequality has been used:

$$\|u\|_{L_p}\leq C\|u_{xxx}\|_{L_2}^a\|u\|_{L_2}^{1-a},\quad ap=\frac{p-1}{6},\quad 1<p<7,$$

From Lemma 1, it follows

$$a\|v_x\|_{L_2}^2\leq\frac{\beta}{3}\|v_{xx}\|_{L_2}^2+C\|v\|_{L_2}^2.$$

Hence from (2.1) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L_2}^2+\frac{\beta}{3}\|v_{xx}\|_{L_2}^2\leq(C_2+b)\|v\|_{L_2}^2+C_3.$$

The Gronwall's inequality implies (2.6)

Corollary 1

$$\sup_{0\leq t\leq T}\|u\|_{L_\infty(R^1)}\leq E_2 \tag{2.8}$$

where the constant E_2 only depends on $\|u_0\|_{H^1(R^1)}$.

Lemma 5 Suppose that the conditions of Lemma 4 are satisfied, and assume that

(1) $f(u)\in C^{m+2}$, $g(u)\in C^{m+1}$, $m\geq 1$;

(2) $u_0(x)\in H^{m+1}(R^1)$

Then for the solution of problem (1.3)(1.4) there is a estimation

$$\sup_{0\leq t\leq T}\|u(\cdot,t)\|_{H^{m+1}(R^1)}\leq E_m, \tag{2.9}$$

where the constant E_m only depends on $\|u_0\|_{H^m}$.

Proof Obvionsly, the lemma is true as $m=0$. Now suppose $\|u(\cdot,t)\|_{H^m(R^1)}\leq E_m$ $(m\geq 1)$. Differentiating (2.1) with respect to x for $m+1$ times, letting $D_x^{m+1}u=v$ and taking the inner product for the resulting equation with v, it follows

$$(v_t + D_x^{m+2} f(u) + a v_{xx} + \beta v_{xxxx} - D_x^{m+1} g(u), v) = 0. \tag{2.10}$$

Since

$$(v, v_t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L_2}^2, \qquad (v, a v_{xx}) = -a\|v_x\|_{L_2}^2,$$

$$(\beta v_{xxxx}, v) = \beta\|v_{xx}\|_{L_2}^2, \qquad (v, D_x^{m+2} f(u)) = -(v_x, D_x^{m+1} f(u)).$$

Because the following inequality [7] holds:

$$\|f(u(\cdot,t)\|_{H^{m+1}(R^1)} \leq C M_m(f_0, v_0)(1 + \|u(\cdot,t)\|_{H^m(R^1)})^{m+1}\|u(\cdot,t)\|_{H^{m+1}(R^1)} \tag{2.11}$$

where $f_0 = \max\limits_{s\leq m+1}\sup\limits_{|u|\leq v_0}|D^s f(u)|$, $|u| \leq U_0 = \sup\limits_{\tau}\|u(\cdot,\tau)\|_{L_\infty(R^1)}$, the constant M_m depends on the constants m, f_0 and U_0.

Thus it follows

$$|(v, D_x^{m+2} f(u))| \leq \|v_x\|_{L_2}\|D_x^{m+1} f(u)\|_{L_2} \leq C\|D_x^{m+1} u\|_{L_2}\|u\|_{H^{m+1}} \leq \frac{\beta}{3}\|D^{m+3} u\|_{L_2}^2 + C.$$

Similarly, we have

$$|(v, D_x^{m+1} g(u))| \leq C_2\|v\|_{L_2}\|u\|_{H^{m+1}} \leq \frac{\beta}{3}\|D_x^{m+3} u\|_{L_2}^2 + C_3$$

and

$$a\|v_x\|_{L_2}^2 \leq \frac{\beta}{3}\|v_{xx}\|_{L_2}^2 + C_4\|v\|_{L_2}^2.$$

Hence from (2.10) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L_2}^2 + \frac{\beta}{3}\|v_{xx}\|_{L_2}^2 \leq C_4\|v\|_{L_2}^2 + C_5.$$

The Gronwall's inequality implies the conclusions of the lemma.

Corollary 2 $m \geq 3$, we have

$$\|u_t\|_{L^\infty(0,T;\,H^2)} \leq E_4 \tag{2.12}$$

where the constant E_4 depends on $\|u_0\|_{H^4(R^1)}$. (2.12)

Proof It is easy to see that from the equation (1.3).

§ 3 The Existence and Uniqueness of the Global Solution for Problem (1.3) (1.4)

The existence of the local solution for problem (1.3) (1.4) will be proved firstly. Then applying the priori estimations in § 2 and the continuation extension principle we can prove the existence of the global solution for problem (1.3) (1.4). For the purpose, one needs some lemmas.

Lemma 6 Suppose $u_0(x) \in H^s(R^1)$. Then for the sloution $\varphi(x,t)$ of problem (3.1) (3.2)

$$\varphi_t + a\varphi_{xx} + \beta\varphi_{xxxx} = 0, \quad a > 0, \ \beta > 0 \tag{3.1}$$

$$\varphi|_{t=0} = \varphi_0(x). \tag{3.2}$$

We have the estimation

$$\|\varphi(\cdot,t)\|_{H^s} \leq C\|u_0(x)\|_{H^s}, \tag{3.3}$$

where the constant C is independent of φ.

Proof Obviously, the solution of problem (3.1) (3.2) can be express as the following form

$$\varphi(x,t)=E(x,t)*u_0(x) \tag{3.4}$$

where $E(x,t)=\int_{-\infty}^{\infty}\exp(-i\xi x-\beta\xi^4 t+a\xi^2 t)\mathrm{d}\xi$

is a foundational solution of equation (3.1), "$*$" denotes the convolution for x. From (3.5) it follows

$$\begin{aligned}\|D^s\varphi(\cdot,t)\|_{L_2}&=\|(i\xi)^s\hat{u}_0(\xi)\hat{E}(\xi,t)\|_{L_2}\\&=\|(i\xi)^s\hat{u}_0(\xi)e^{-t(\beta\xi^4-a\xi^2)}\|_{L_2}\\&\leq\int_{-\infty}^{\infty}(1+|\xi|^2)^s|\hat{u}_0(\xi)|^2e^{-2t(\beta\xi^4-a\xi^2)}\mathrm{d}\xi\ ,\quad t\geq 0\\&=\int_{|\xi|\leq\sqrt{a/\beta}}+\int_{|\xi|>\sqrt{a/\beta}}=I_1+I_2\end{aligned}$$

$$I_1\leq\int_{-\infty}^{\infty}(1+|\xi|^2)^s|\hat{u}_0(\xi)|^2e^{2t\frac{a^2}{4\beta}}\mathrm{d}\xi=e^{\frac{a^2}{2\beta}T}\|u_0(x)\|_{H^s}=C\|u_0\|_{H^s}$$

$$I_2\leq\int_{\xi>\sqrt{a/\beta}}(1+|\xi|^2)^s|\hat{u}_0(\xi)|^2\mathrm{d}\xi\leq\|u_0(x)\|_{H^s}.$$

Thus we have

$$\|D^s\varphi(\cdot,t)\|_{L_2}\leq e^{\frac{a^2}{2\beta}T}\|u_0(x)\|_{H^s}=C\|u_0(x)\|_{H^s}.$$

Lemma 7 Suppose $a(x,t)\in L^\infty(0,T;H^{s-3})$, $s\geq 3$. Then for the solution $w(x,t)$ of problem (3.5) (3.6)

$$w_t+aw_{xx}+\beta w_{xxxx}=a(x,t) \tag{3.5}$$

$$w(x,0)=0, \tag{3.6}$$

we have the estimation

$$\|w(\cdot,t)\|_{H^l}\leq C(a,\beta)A(t)\sup_{0\leq\tau\leq t}\|a(\cdot,\tau)\|_{H^{l-3}}\ ,\quad l=3,4,\cdots,s, \tag{3.7}$$

where the constant $C(a,\beta)>0$, $A(t)=c_1e^{\frac{a^2}{2\beta}T}+c_2(t^{\frac{1}{4}}+t^{\frac{1}{2}}+t^{\frac{3}{4}}+t)$, $c_1,c_2>0$. For $l=0,1,2,3$ the lemma is also true.

Proof The solution of problem (3.5) (3.6) can be express as follows

$$w(x,t)=\int_0^t E(x,t-\tau)*a(x,\tau)\mathrm{d}\tau$$

in the sense of distribution. By using the properties of Fourier transformation and the estimations, it follows

$$\|D^k w(\cdot,t)\|_{L_2}\leq\int_0^t\|(i\xi)^k\exp(-\beta\xi^4+a\xi^2)(t-\tau)a(\xi,\tau)\|_{L_2}\mathrm{d}\tau\ ,$$

where

$$\|(i\xi)^k\exp(-\beta\xi^4+a\xi^2)(t-\tau)a(\xi,\tau)\|_{L_2}^2$$

$$\leq\int_{-\infty}^{\infty}(1+|\xi|^2)^k e^{-2(\beta\xi^4-a\xi^2)(t-\tau)}|\hat{a}(\xi,\tau)|^2 d\xi$$

$$\leq\int_{|\xi|\leq\sqrt{a/\beta}}(1+|\xi|^2)^k e^{-2(\beta\xi^4-a\xi^2)(t-\tau)}|\hat{a}(\xi,\tau)|^2 d\xi+\int_{|\xi|>\sqrt{a/\beta}}$$

$$=I_1+I_2,$$

$$|I_1|\leq\int_{|\xi|\leq\sqrt{a/\beta}}(1+|\xi|^2)^3 e^{-2(\beta\xi^4-a\xi^2)(t-\tau)}(1+|\xi|^2)^{k-3}|\hat{a}|^2 d\xi$$

$$\leq\int_{|\xi|\leq\sqrt{a/\beta}}(1+|\xi|+|\xi|^2+|\xi|^3)^2 e^{\frac{a^2}{2\beta}(t-\tau)}(1+|\xi|^2)^{k-3}|\hat{a}|^2 d\xi$$

$$\leq[1+\sqrt{a/\beta}+a/\beta+(a/\beta)^{\frac{3}{2}}]e^{\frac{a^2}{2\beta}(t-\tau)}\|a(\cdot,\tau)\|_{k-3}^2$$

$$|I_2|\leq\int_{|\xi|>\sqrt{a/\beta}}(1+|\xi|+|\xi|^2+|\xi|^3)^2 e^{-\beta\xi^4(t-\tau)}(1+|\xi|^2)^{k-3}|\hat{a}|^2 d\xi$$

$$\leq\int_{-\infty}^{\infty}[\sum_{l=0}^{3}(\beta(t-\tau))^{-\frac{l}{4}}z^l]^2 e^{-z^4}(1+|\xi|^2)^{k-3}|\hat{a}|^2 d\xi$$

$$\leq C\int_{-\infty}^{\infty}[\sum_{\tau=0}^{3}(\beta(t-\tau))^{-\frac{l}{4}}]^2(1+|\xi|^2)^{k-3}|a|^2 d\xi,$$

where $z=(\beta(t-\tau))^{\frac{1}{4}}|\xi|$, $z^{2l}e^{-z^4}\leq\text{const}$, $i=0,1,2,3$. Hence we have

$$\|D^k w(t)\|_{L_2}\leq C_1 e^{\frac{a^2}{2\beta}t}t\sup_{0\leq\tau\leq t}\|a(\cdot,\tau)\|_{k-3}$$

$$+C_2(t^{\frac{1}{4}}+t^{\frac{1}{2}}+t^{\frac{3}{4}}+t)\sup_{0\leq\tau\leq t}\|a(\cdot,\tau)\|_{k-3}$$

$$=A(t)\sup_{0\leq\tau\leq t}\|a(\cdot,\tau)\|_{H^{k-3}},$$

where

$$C_1=1+(a/\beta)^{\frac{1}{2}}+a/\beta+(a/\beta)^{\frac{3}{2}},$$

$$A(t)=C_1 e^{\frac{a^2}{2\beta}t}t+C_2(t^{\frac{1}{4}}+t^{\frac{1}{2}}+t^{\frac{3}{4}}+t).$$

The lemma is proved.

Now we define the operator of the solution $u(x,t)$ of problem (1.3) (1.4) as follows

$$J[u]=E(x,t)*u_0(x)+\int_0^t E(x,t-\tau)*[-f(u)_x+g(u)]d\tau, \tag{3.8}$$

where $E(x,t)$ is a foundational solution of equation (3.1).

Let set

$$B=\{u(x,t)\in L^\infty(0,T;H^s),\quad \|u\|_{L^\infty(0,T;H^s)}\leq M\},$$

where $M\geq 2e^{\frac{a^2}{2\beta}T}\|u_0\|_{H^s}$, $s\geq 1$, and $d(u,v)=\|u-v\|_{H^s}$, $u,v\in B$.
Then set B is a complete space.

Lemma 8 Suppose that $f(u)\in C^{s+1}$, $g(u)\in C^{s}$, and $u_0(x)\in H^{s}$. Then the mapping (3.8) has the following properties:

(1) $$\|J[u]-J[v]\|_B\le CA(t)\|u-v\|_B,\quad \forall u,v\in B \tag{3.9}$$

where the constant C is independent of u, v, only depends on a, β, s.

(2) $\|J[u]\|_B\le M$, when t is suitably small.

Proof From (3.8), it follows

$$J[u]-J[u]=\int_0^t E(x,t-\tau)*[-D_x(f(u)-f(v))+g(u)-g(v)]d\tau$$

lamma 6 and lemma 7 implies that

$$\|J[u]-J[v]\|_B\le C(a,\beta)A(t)\sup_{0\le\tau\le t}\|-D_x(f(u)-f(v))+g(u)-g(v)\|_{H^{s-3}}$$

where $A(t)=C_1e^{\frac{a^2}{2\beta}t}t+C_2(t^{\frac{1}{4}}+t^{\frac{1}{2}}+t^{\frac{3}{4}}+t)$. By using the Leibniz principle, Hölder inequality and inequality (2.11) it follows

$$\|-D_x(f(u)-f(v))+g(u)-g(v)\|_{H^{s-3}}\le C_3\|u-v\|_{H^{s-2}}$$

where the constant C_3 depends on $\sup\limits_{\substack{k\le s+1\\|u|\le v_0}}|D^kf(u)|$, $\sup\limits_{\substack{k\le s\\|u|\le v_0}}|D^kg(u)|$, and

$$U_0=\sup_{0\le\tau\le t}\|u(\cdot,\tau)\|_{L_\infty(R^1)}.$$

Hence we get

$$\|J[u]-J[v]\|_B\le C(a,\beta)C_3A(t)\|u-v\|_{H^{s-2}}\le CA(t)\|u-v\|_B.$$

This implies (3.9). Then from (3.8), Lemma 6 and Lemma 7 it follows

$$\begin{aligned}\|J[u]\|_B&\le e^{\frac{a^2}{2\beta}T}\|u_0(x)\|_{H^s}+C(a,\beta)A(T)\|u\|_{H^{s-1}}\\&\le e^{\frac{a^2}{2\beta}T}\|u_0(x)\|_{H^s}+C(a,\beta)A(T)\|u\|_{H^s}\\&\le\frac{1}{2}M+C(a,\beta)A(T)M\le M\end{aligned}$$

where $C(a,\beta)\ A(T)<\frac{1}{2}$ as T is suitably small. The lemma is proved.

Theorem 1 Supose $a>0$, $\beta>0$ and assume that the following conditions are satisfied:

(1) $f(u)\in C^{s+1}$, $g(u)\in C^{s}$;

(2) $u_0(x)\in H^{s}$, $s\ge1$

Then there exists a local solution $u(x,t)$ of problem (1.3) (1.4), $u(x,t)\in L^\infty(0,T_0;H^s(R^1))$, where the constant T_0 depends on $\|u_0\|_{H^s}$, a, β and s.

Proof From lemma 8 we known that the $J[u]$ is contracted and $J[u]$ map into inself as t is suitably small. By using the fixed point principle, the mapping possesses a fixed point $u=J[u]$ and it is a solution satisfying the integral equation

$$u(x,t)=E(x,t)*u_0(x)+\int_0^t E(x,t-\tau)*[-f(u)_x+g(u)]d\tau. \tag{3.10}$$

It is easy to know [8] that $J[u]$ is strong differentiable with respect to t, then local solution of problem (2.1) (2.2) can be obtained.

Theorem 2 Suppose $a>0$, $\beta>0$ and assume that the following conditions are satisfied:

(1) $f(u)\in C^{s+1}$, $|f(u)|\leq A|u|^p$, $1<p<7$, $A=\text{const}>0$;

(2) $g(u)\in C^s$, $g'(u)\leq b$, $g(0)=0$, $b=\text{const}>0$;

(3) $u_0(x)\in H^s(R^1)$, $s\geq 1$.

Then there exists a global smooth solution $u(x,t)$ of problem (1.3) (1.4), $u(x,t)\in L^\infty(0,T;H^s(R^1))$.

Proof From From theorem 1 there exists a local smooth solution $u(x,t)$ of problem (1.3) (1.4), $u(x,t)\in L^\infty(0,T_0;H(R^1))$, where T_0 is only depends on $\|u_0\|_{H^s(R^1)}$. Because the priori estimations in § 2 hold, and by using the continuation extension principle, we can get the global smooth solution $u(x,t)$ of problem (1.3) (1.4), $u(x,t)\in L^\infty(0,T;H^s(R^1))$.

Theorem 3 Suppose that $f(u)\in C^2$, $g(u)\in C^1$. Then the smooth solution of problem (1.3)(1.4) is unique.

Proof Suppose that there are two solution of problem (1.3)(1.4) $u(x,t)$ and $v(x,t)$. Letting $w(x,t)=u(x,t)-v(x,t)$, then from (1.3) (1.4) we have

$$w_t+aw_x+\beta w_{xxxx}+(f(u)-f(v))_x=g(u)-g(v) \tag{3.11}$$

$$w|_{t=0}=0,\quad x\in R^1 \tag{3.12}$$

where

$$f(u)-f(v)=\int_0^1 f_u'(\tau u+(1-\tau)v)\mathrm{d}\tau w,$$

$$g(u)-g(v)=\int_0^1 g_u'(\tau u+(1-\tau)v)\mathrm{d}\tau w.$$

Taking the inner product for (3.11) with w, it follows

$$(w_t+aw_{xx}+\beta w_{xxxx}+(w\int_0^1 f_u'\mathrm{d}\tau)_x-w\int_0^1 g_u'\mathrm{d}\tau, w)=0. \tag{3.13}$$

Since

$$|(w,\int_0^1 g_u'\mathrm{d}\tau w)|\leq\|\int_0^1 g_u'\mathrm{d}\tau\|_{L_\infty}\|w\|_{L_2}^2\leq C\|w\|_{L_2}^2$$

$$|((w\int_0^1 f_u'\mathrm{d}\tau)_x,w)|=|(w_x,w\int_0^1 f_u'\mathrm{d}\tau)|\leq a\|w_x\|_{L_2}^2+C\|w\|_{L_2}^2.$$

$$2a\|w_x\|_{L_2}^2\leq C_1\|w\|_{L_2}^2+\frac{\beta}{2}\|w_{xx}\|_{L_2}^2$$

(3.13) impplies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L_2}^2+\frac{\beta}{2}\|w_{xx}\|_{L_2}^2\leq C_2\|w\|_{L_2}^2 .$$

By using Gronwall's inequality, it follows $w\equiv 0$. The theorem is proved.

§ 4 The Existence of the Global Smooth Solution of Problem (1.5) (1.6)

Now we consider the initial value problem for the systems of generalized KS type equations in n dimensions

$$\boldsymbol{u}_t + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \operatorname{grad}\varphi(u_1, \cdots, u_N) + a\Delta \boldsymbol{u} + \beta\Delta^2 \boldsymbol{u} = \boldsymbol{g}(\boldsymbol{u}) \tag{1.5}$$

$$\boldsymbol{u}\big|_{t=0} = \boldsymbol{u}(x) \tag{1.6}$$

where $\boldsymbol{u}(x, t) = (u_1(x,t), \cdots, u_N(x, t))$, $a>0$, $\beta>0$. By using the similar method, Sobolev's estimation (2.5), and [8, § 10, Lemma 7], the following theorem can be obtained:

Theorem 4 Suppose that the following conditions are satisfied:

(1) $\varphi(\boldsymbol{u}) \in C^{s+2}$, $|\operatorname{grad}\varphi(\boldsymbol{u})| \le A|\boldsymbol{u}|^p$, $p<\frac{6}{n}+1$, $A=\text{const}>0$;

(2) $\boldsymbol{g}(\boldsymbol{u}) \in C^s$, The Jacobi derivative matrix $\boldsymbol{g}_u(\boldsymbol{u})$ is semibounded, i.e., there is a constant b, such that

$$\boldsymbol{\xi} \cdot \boldsymbol{g}_u(\boldsymbol{u})\boldsymbol{\xi} \le b|\boldsymbol{\xi}|^2$$

for any $\xi \in R^n$, where "$\cdot$" denotes the scalar product operator of two N-dimensional vectors and

(3) $\boldsymbol{u}_0(x) \in H^s(R^n)$, $s \ge [\frac{n}{2}]+1$.

Then there exists a global smooth solution $u(x, t)$ of problem (1.5) (1.6), $\boldsymbol{u}(x, t) \in L^\infty(0, T; H^s(R^n))$.

The initial-boundary value problem for the system of equations (1.5), i.e.,

$$\boldsymbol{u}_t + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \operatorname{grad}\varphi(\boldsymbol{u}) + a\Delta \boldsymbol{u} + \beta\Delta^2 \boldsymbol{u} = \boldsymbol{g}(\boldsymbol{u}) \tag{4.1}$$

$$\boldsymbol{u}\big|_{\partial_n} = 0 \tag{4.2}$$

$$\boldsymbol{u}\big|_{t=0} = \boldsymbol{u}(x) \tag{4.3}$$

where $\Omega \subset R^n$ is a bounded domain, its boundary $\partial\Omega \in C^2$. By using Galerkin method, choosing the basic functions $\{w_j(x)\}$ as follows

$$-\Delta w_j = \lambda_j w_j, \quad w_j\big|_{\partial n} = 0 \tag{4.4}$$

we can prove the following theorem:

Theorem 5 If the conditions of Theorem 4 are satisfied, and $\boldsymbol{u}_0(x) \in H^s(\Omega) \cap H_0^1(\Omega)$. Then there exists a global smooth solution $u(x, t)$ of problem (4.1)—(4.3),

$$\boldsymbol{u}(x, t) \in L^\infty(0, T; H^s(\Omega) \cap H_0^1(\Omega)).$$

§ 5 The "Blowing up" problem for Some Dissipative Equations

In this section, first we consider the "blowing up" problem for the following dissipative equatione

$$\begin{cases} u_t + a u_{xx} + \beta u_{xxxx} + cu = g(u) & (5.1) \\ u|_{t=0} = u_0(x), \quad x \in R^1 & (5.2) \end{cases}$$

where the constant $a>0$, $\beta>0$, and $C>0$. Suppose that a local smooth solution of problem (5.1) (5.2)) exists.

One can prove that under some conditions the solution of problem (5.1) (5.2) "blow up", i.e., there is $T_0>0$, such that for the solution $u(x,t)$ of problem (5.1) (5.2),

$$\lim_{t\to T_0^-} \sup(u(t),\ u(t)) = +\infty \tag{5.3}$$

Theorem 6 Suppose that there exists a smooth solution $u(x,t)$ of problem (5.1) (5.2), and assume that

(1) $2\beta \geq a > 0$, $C \geq \beta > 0$;

(2) $g(0)=0$, and there is a constant $\delta>0$, such that

$$2(\delta+1)G(u) \leq (u, g(u)) \tag{5.4}$$

where $G(u) = \int_0^1 (g(\rho u), u)\,d\rho$;

(3) The initial value function $u_0(x)$ satisfies

$$G(u_0) > \frac{1}{2}[C(u_0, u_0) - a(u_{0x}, u_{0x}) + \beta(u_{0xx}, u_{0xx})] \tag{5.5}$$

Then the solution of problem (5.1) (5.2) "blow up", i.e., there is a constant

$$T \leq T_{r_0\delta} = [\frac{(2\delta+1)(u_0, u_0)}{2\delta^2(\delta+1)}]\{G(u_0) - \frac{1}{2}[C(u_0,u_0) - a(u_{0x}, u_{0x}) + \beta(u_{0xx}, u_{0xx})]\}^{-1} < \infty, \tag{5.6}$$

such that

$$\lim_{t\to T_{r_0\delta}^-} \sup_{0\leq\tau\leq t} (u(\cdot,\tau), u(\cdot,\tau)) = +\infty. \tag{5.7}$$

Proof The method of convex function can be used. For any $T_0>0$, $r>0$, and $\tau>0$, let

$$F(t) = \int_0^t (u, u)\,d\eta + (T_0 - t)(u_0, u_0) + r(t+\tau)^2 \tag{5.8}$$

$t\in[0, T_0]$. Since

$$F'(t) = (u,u) - (u_0, u_0) + 2r(t+\tau)$$
$$= 2\int_0^t (u, u_\eta)\,d\eta + 2r(t+\tau). \tag{5.9}$$

From (5.8), (5.9), $F'(0) = 2r > 0$ and $F(t)>0$, $\forall t\in[0, T_0]$. Thus $F^{-\delta}(t)$ is definite, as any $\delta>0$. If we can prove

$$(F^{-\delta}(t))'' \leq 0 \tag{5.10}$$

and $(F^{-\delta}(0))' < 0$, then one have

$$F^{-\delta}(t) \leq F^{-\delta}(0) + [F^{-\delta}(0)]'t \quad \text{or} \quad F(t) \leq F^{1+\delta}(0)\{F(0) - \delta t F'(0)\}^{-\frac{1}{\delta}}.$$

So the solution of problem (5.1) (5.2) "blow up", $F(t)\to\infty$ as $t\to F(0)/\delta F'(0)$.

It is easy to know that the following inequality

$$F(t)F''(t)-(\delta+1)F'(t)\geq 0 \tag{5.11}$$

implies the inequality (5.10). Differentiating $F'(t)$ with respect to t, it follows

$$\begin{aligned}F''(t)&=2\int_0^t(u_\eta,u)_\eta d\eta+2(u_t,u)_0+2r\\&=4(\delta+1)[\int_0^t(u_\eta,u_\eta)d\eta+r]+2\int_0^t[(u_\eta,u))_\eta-\\&-2(\delta+1)(u_\eta,u_\eta)]d\eta+2[(u_t,u)_0-(2\delta+1)r].\end{aligned} \tag{5.12}$$

The equation (5.1) can be written as the following operator form

$$\frac{du}{dt}+Au=g(u) \tag{5.13}$$

where $Au\equiv au_{xx}+\beta u_{xxxx}+Cu$. Hence from (5.8) (5.9) and (5.12) (5.13), and by proceeding the direct calaulations it follows

$$\begin{aligned}FF''-(\delta+1)F'^2&=4(\delta+1)S^2+2F\{-\int_0^t[(u,Au)_\eta-2(\delta+1)(u_\eta Au)]d\eta\\&+2F\int_0^t[(u,g(u))_\eta-2(\delta+1)(u_\eta,g(u))]d\eta\\&+4(\delta+1)(T_0-t)(u_0,u_0)[\int_0^t(u_\eta,u_\eta)d\eta+r]\\&+2F[(u_t,u)_0-(2\delta+1)r],\end{aligned} \tag{5.14}$$

where

$$\begin{aligned}S^2&=[\int_0^t(u,u)d\eta+r(t+\tau)^2][\int_0^t(u_\eta,u_\eta)d\eta+r]\\&-[\int_0^t(u_\eta,u)d\eta+r(t+\tau)]^2\geq 0.\end{aligned}$$

By the direct verification, it is easy to get

$$\frac{d}{dt}G(v(t))=(g(v(t)),v_t(t)). \tag{5.15}$$

Thus from (5.14) (5.15) we have

$$\begin{aligned}FF''-(\delta+1)F'^2&\geq 4\delta F\int_0^t(u_\eta,Au)d\eta+2F[(u,g(u))-2(\delta+1)G(u)]\\&+2F[2(\delta+1)G(u_0)-(u_0,Au_0)-(2\delta+1)r]\end{aligned} \tag{5.16}$$

where

$$2(u_t,Au)=\frac{d}{dt}(u,Au)$$

$$FF''-(\delta+1)F'^2\geq 2\delta F(u,Au)+4(\delta+1)F[G(u_0)-\frac{1}{2}(u_0,Au_0)-(2\delta+1)\frac{r}{2(\delta+1)}]$$

$$\begin{aligned}(u,Au)&=-a\|u_x\|_{L_2}^2+\beta\|u_{xx}\|_{L_2}^2+C\|u\|_{L_2}^2\\&\geq -a\|u_x\|_{L_2}^2+2\beta\|u_x\|_{L_2}^2-\beta\|u\|_{L_2}^2+C\|u\|_{L_2}^2\geq 0.\end{aligned}$$

If $r_0=\frac{2(\delta+1)}{(2\delta+1)}(G(u_0)-\frac{1}{2}(u_0,Au_0))>0$, $T_{r_0\tau}=\frac{F(0)}{\delta F'(0)}=\frac{T_0(u_0,u_0)+r_0\tau^2}{2r_0\delta\tau}$, and taking $T_0=T_{r_0\tau}$, then $T_{r_0\tau}=r_0\tau^2/(2\delta r_0\tau-(u_0,u_0))=f(\tau)$. One finds that $f(\tau)$ has a minimun at $\tau=(u_0,u_0)/\delta r_0$, and this minimun is $(u_0,u_0)/\delta^2 r_0$. So that

$$T_{r_0\delta}=[\frac{(2\delta+1)(u_0,u_0)}{2\delta^2(\delta+1)}]\{G(u_0)-\frac{1}{2}[C(u_0,u_0)+\beta(u_{0xx},u_{0xx})-\alpha(u_{0x},u_{0x})]\}^{-1}$$

Hence

$$FF''-(\delta+1)(F')^2\geq 0, \qquad \lim_{t\to T_{r_0\delta}}\sup(u(t),u(t))=+\infty.$$

The theroem is proved.

Now we consider the "blowing up" problem of the following dissipative equation with initial condition

$$\begin{cases} u_t+au_{xx}+f(u)_x=g(u) & (5.17)\\ u|_{t=0}=u_0(x), \quad x\in R^1. & (5.18)\end{cases}$$

We have

Theorem 7 Suppose that there exists the smooth solution of problem (5.17)(5.18), and assume that the following conditions are satisfied:

(1) $a>0$;

(2) $(u,g(u))\geq C(u,u)^{1+\delta}$, $C=\text{const}>0$, $\delta>0$;

(3) $\|u_0(x)\|_{L_2(R^1)}>0$.

Then the solution of problem (5.17)(5.18) "blow up", i.e., there is $T_0>0$, such that for the solution $u(x,t)$ of problem (5.15)(5.16)

$$\lim_{t\to T_0^-}\sup(u(t),u(t))=+\infty.$$

Proof Taking the inner product for (5.17) with $u(x,t)$, it follows

$$(u_t+au_{xx}+f(u)_x,u)=(g(u),u). \tag{5.19}$$

Since

$$(u,u_t)=\frac{1}{2}\frac{d}{dt}\|u\|^2_{L_2}, \quad (u,au_{xx})=-a\|u_x\|^2_{L_2}, \quad (u,f(u)_x)=0.$$

Hence from (5.19) it follows

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L_2}\geq a\|u_x\|^2_{L_2}+C(u,u)^{1+\delta}\geq C(u,u)^{1+\delta}, \quad (\delta>0).$$

It is easy to see that

$$\|u(\cdot,t)\|^2_{L_2}\to+\infty, \quad t\to t_0<\infty.$$

The theorem is proved.

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广义Kuramoto-Sivashinsky型方程初值问题光滑整体解的存在性与破裂性

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摘 要

本文用积分估计和不动点原理证明了一类广义 Kuramoto-Sivashinsky 型方程初值问题光滑整体解的存在唯一性，并在不同情况下讨论了解的破裂性，并给出了导致破裂性的充分条件.