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Globally Asymptotic Behaviour of a Class of Integro-differential Equations*

Hu Shigeng

(Huazhong University of Science and Technology, Wuhan)

§ 1 Introduction

A number of problems in ecology and economics leads us to consider the integro-differential equation

$$\dot{x}(t)=\operatorname{diag}(x(t))\{a+\operatorname{diag}(b)x(t)+\int_{-r}^{0}[d\mu(s)]x(t+s)\},\tag{1}$$

where $\operatorname{diag}(x(t))=\operatorname{diag}(x_1(t),\cdots,x_n(t))$, $a,b\in\mathbf{R}^n$, $r\in\mathbf{R}_+$, μ is an $\mathbf{R}^{n\times n}$-valued measure on $[-r,0]$, $\mathbf{R}^{n\times n}=L(\mathbf{R}^n)$. The equation (1) may be considered in the setting of the functional differgntial equation

$$\dot{x}(t)=\operatorname{diag}(x(t))g(t,x_t),\tag{2}$$

where x_t denotes the function $s\mapsto x(t+s)$ on $[-r,0]$. As $r=0$, the equation (2) becomes the well-known Kolmogorov model in ecology. Smith[8] studied the equation

$$\dot{x}(t)=f(t,x_t)\tag{3}$$

under some assumption of monotonicity about f, thus, in some extent, generalized the monotone flow theory of Hirsch[2]. In [3], we generalized the results of Smith[8] and gave some criteria to determine global asymptotic stability of functional differential equations with certain monotonicity property. In this paper, we apply the results and methods of [8] and [3] to explain the asymptotic behaviour of the equation (1).

§ 2 Preliminaries

Let C denote the Banach space $C([-r,0],\mathbf{R}^n)$ with the sup norm. In $\mathbf{R}^n$ ($\mathbf{R}^{n\times n}$ or C, resp.) we use the order $\leq$ induced by the cone $\mathbf{R}^n_+$ ($\mathbf{R}^{n\times n}_+$ or $C_+=C([-r,0],\mathbf{R}^n_+)$, resp.). We write $x<y$ iff $y-x\in\mathring{\mathbf{R}}^n_+$ ($\mathring{\mathbf{R}}^n_+$ denotes the interior of $\mathbf{R}^n_+$) and $\varphi<\psi$ iff $\psi-\varphi\in C^\circ_+$.

First consider the equation (3). Suppose that $\Omega\subset\mathbf{R}\times C$ is an open set on which $f(t,\varphi)$ and $D_\varphi f(t,\varphi)$ are continuous. Then (3) has exactly one solution

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$x=x(t,\sigma,\varphi)$ through every $(\sigma,\varphi)\in\Omega$[1], denote this solution also by $x(\sigma,\varphi)$ or $x(\sigma,\varphi,f)$.

Definition 1 If for any $(t,\varphi)\in\Omega$, $u\in C_+$, $u_i(0)=0$ for some i implies $D_\varphi f_i(t,\varphi)u\geq 0$, then f is said to be cooperative.

Lemma 1 Suppose $f(t,\varphi)\leq\overline{f}(t,\varphi)$ for all $(t,\varphi)\in\Omega$, f or $\overline{f}$ is cooperative. For given (σ,φ), $(\sigma,\psi)\in\Omega$, if $\varphi\leq\psi$, then $x_t(\sigma,\varphi,f)\leq x_t(\sigma,\psi,\overline{f})$ for such $t\geq\sigma$ at which both $x_t(\sigma,\varphi,f)$ and $x_t(\sigma,\psi,\overline{f})$ are defined.

This is an immediate consequence of Proposition 1.1 in [8].

Below we suppose

$$f(t,\varphi)=\mathrm{diag}(\varphi(0))g(t,\varphi),\quad (t,\varphi)\in\Omega, \tag{4}$$

both g and $D_\varphi g$ are continuous on Ω. An easy calculation gives

$$D_\varphi f_i(t,\varphi)u=\varphi_i(0)D_\varphi g_i(t,\varphi)u+u_i(0)g_i(t,\varphi),\quad i=1,2,\cdots,n. \tag{5}$$

(5) together with Definition 1 imply that, if $\Omega\subset\mathbf{R}\times C_+$, then f is cooperative iff g is cooperative. In the case that g is cooperative, Lemma 1 implies (put $f=\overline{f}$) $x_t(\sigma,\varphi)\leq x_t(\sigma,\psi)$ for (σ,φ), $(\sigma,\psi)\in\Omega$, with $0\leq\varphi\leq\psi$ and $t\geq\sigma$, whenever both $x_t(\sigma,\varphi)$ and $x_t(\sigma,\psi)$ are defined. In particular, $x_t(\sigma,\varphi)\geq 0$ for $\varphi\geq 0$ and $t\geq\sigma$, since $x(t)\equiv 0$ satisfies the equation (2).

Lemma 2 Let $x=x(t)$ be a solution of (2) defined on $[\sigma,\rho)$. If $x_i(t_0)>0$ ($x_i(t_0)<0$, resp.) for some $t_0\in[\sigma,\rho)$ and some index i, then $x_i(t)>0$ ($x_i(t)<0$, resp.) for all $t\in[\sigma,\rho)$. Hence $x(t_0)>0$ for some $t_0\in[\sigma,\rho)$ implies $x(t)>0$ on the global interval $[\sigma,\rho)$.

Proof It suffices to consider the case that $x_i(t_0)>0$. By the continuity of $x(t)$, there exists a maximal $\tau\in(t_0,\rho]$ such that $x_i(t)>0$ for all $t\in[t_0,\tau)$. We claim $\tau=\rho$. For if $\tau<\rho$, then $x_i(\tau)=0$, so that

$$\int_{t_0}^{\tau}g_i(t,x_t)\,\mathrm{d}t=\int_{t_0}^{\tau}\frac{\dot{x}_i(t)}{x_i(t)}\mathrm{d}t=\lim_{t\uparrow\tau}\ln\frac{x_i(t)}{x_i(t_0)}=-\infty,$$

this contradicts the continuity of $g_i(t,x_t)$. Therefore $x_i(t)>0$ on $[t_0,\rho)$. Similarily we have $x_i(t)>0$ for all $t\in[\sigma,t_0)$.

Lemma 3 Let $f:C\to\mathbf{R}^n$ be a completely continuous C^1 map, and let $x(t)=x(t,0,\varphi)$ be a solution of the equation

$$\dot{x}(t)=f(x_t). \tag{6}$$

If $x(t)$ assumes its values in a fixed compact subset of $\mathbf{R}^n$, then $x(t)$ is defined on $[0,\infty)$. If, in addition, $x(t_k)\to\overline{x}$ for some sequence $t_k\to\infty$ as $k\to\infty$, then (6) has a bounded solution $y(t)$ on $\mathbf{R}$ with $y(0)=\overline{x}$.

Proof The former conclusion follows immediately from [1, ch.2, Th.3.2]. To prove the latter, let $x^k(t)=x(t_k+t)$ for $t\geq -t_k$ and $k=1,2,\cdots$, then the sequence $\{x^j\mid j\geq k\}$ is uniformly bounded and equicontinuous on $[-t_k,\infty)$. Hence we may assume that $\{x^k\}$ compact uniformly converges to a continuous function

$y(t)$ on $\mathbf{R}$ as $k\to\infty$. Clearly $y(0)=\overline{x}$ and x^k satisfies

$$x^k(t)=x(t_k)+\int_0^t f(x_\tau^k)\mathrm{d}\tau,\ t\geq -t_k,\ k=1,2,\cdots. \tag{7}$$

Moreover, $\|x_t^k-y_t\|\rightrightarrows 0$ as $k\to\infty$ for t in any fixed compact interval. Thus, letting $k\to\infty$ in (7), we obtain

$$y(t)=\overline{x}+\int_0^t f(y_\tau)\mathrm{d}\tau$$

for any $t\in\mathbf{R}$, so that $\dot{y}(t)=f(y_t)$ $(t\in\mathbf{R})$, as required.

§ 3 Main Results

Now consider (1) and the equation

$$\dot{y}(t)=\mathrm{diag}(y(t))\{a+\mathrm{diag}(b)y(t)+\int_{-r}^0[\mathrm{d}\nu(s)]y(t+s)\}, \tag{8}$$

where $\nu=(\nu_{ij})$, $\nu_{ij}=|\mu_{ij}|$ is the variation of the measure μ_{ij} for $i,j=1,2,\cdots,n$ (see [4]), $\mu=(\mu_{ij})$ as in (1). We suppose $\mu_{ii}(\{0\})=0$ $(1\leq i\leq n)$. Let $A=(a_{ij})=\int_{-r}^0\mathrm{d}\nu$ and $B=\mathrm{diag}(b)+A$, then $A\geq 0$ and B is essentially nonnegative, i.e., $\lambda I+B\geq 0$ for some $\lambda\in\mathbf{R}$. For any $\varphi\in C$, we denote the solutions through $(0,\varphi)$ of (1) and (8) by $x(t,\varphi)$ and $y(t,\varphi)$, respectively.

Theorem 1 Suppose $a>0$, $\sum_j a_{ij}<-b_i$ $(i=1,2,\cdots,n)$, $b=(b_i)$. Then B^{-1} exists, $y^*=-B^{-1}a>0$ and $\lim_{t\to\infty}y(t,\varphi)=y^*$ for all $\varphi\in\overset{\circ}{C}_+$.

Proof First note that $b_i<-\sum_j a_{ij}\leq 0$ $(1\leq i\leq n)$, since $A\geq 0$. Let $D=\mathrm{diag}(-b_1^{-1},\cdots,b_n^{-1})$, $M=DA$ and let $\|\cdot\|$ denote the max norm in $\mathbf{R}^n$. Then

$$\|M\|=\max_{\|x\|=1}\|Mx\|=\max_{\|x\|=1,\ 1\leq i\leq n}|\sum_j b_i^{-1}a_{ij}x_j|<1,$$

hence $s(M)\overset{\mathrm{def}}{=\!=}\max\{\mathrm{Re}\,\lambda|\lambda\in\sigma(M)\}<1$. This implies $s(DB)=s(M-I)=s(M)-1<0$. The fact that B is essentially nonnegative and $-b_i^{-1}>0$ $(1\leq i\leq n)$ together with [7,Th.2.3] implies $s(B)<0$, hence B is invertible and $y^*=-B^{-1}a>0$ by [6, th.1.2].

Below fix $\varphi\in\overset{\circ}{C}_+$ and let $y(t)=y(t,\varphi)$, we claim $\lim_{t\to\infty}y(t)=y^*$. We rewrite (8) as $\dot{y}(t)=f(y_t)$, where

$$f(u)=\mathrm{diag}(u(0))(a+Lu); \tag{9}$$

$$Lu=\mathrm{diag}(b)u(0)+\int_{-r}^0[\mathrm{d}\nu(s)]u(s),\quad u\in C.$$

It is easy to see that $f:C\to\mathbf{R}^n$ is completely continuous, since $L:C\to\mathbf{R}^n$ is continuous and linear. Moreover, f is also cooparative since $\nu_{ij}\geq 0$. Let $x\mapsto\hat{x}$ denote the natural embedding from $\mathbf{R}^n$ into C, as in [8], then

$$f(\hat{y})=\mathrm{diag}(y)(a+By),\quad y\in\mathbf{R}^n, \tag{10}$$

so that $f(\hat{y}^*)=0$, i.e., y^* is an equilibrium point of (8). Choose an $\varepsilon>0$ small enough and an $\tau>1$ large enough such that $u^\varepsilon<\varphi<u^\tau$ and $f(u^\varepsilon)=\operatorname{diag}(y^\varepsilon)\cdot(a+By^\varepsilon)>0$, where $u^\varepsilon=\widehat{y^\varepsilon}$, $y^\varepsilon=\varepsilon y^*$. By [3, th.1], $y(t,u^\varepsilon)$ is increasing in $t\geq 0$. The above facts together with Lemma 1 imply

$$\begin{cases} y^\varepsilon\leq y(t,u^\varepsilon)\leq y^*\leq y(t,u^\tau), \\ y(t,u^\varepsilon)\leq y(t)\leq y(t,u^\tau), \end{cases} \tag{11}$$

for those $t\geq 0$ at which all functions in (11) are defined. By (11) and Lemma 3, $y(t,u^\varepsilon)$ is defined and bounded on $[0,\infty)$, hence $p=\lim_{t\to\infty}y(t,u^\varepsilon)$ exists because of its monotonicity. It follows that $f(\hat{p})=0$, and so $p=-B^{-1}a=y^*$ by (10). On the other hand, since

$$f(u^\tau)=\operatorname{diag}(y^\tau)(a+\tau By^*)=(1-\tau)\operatorname{diag}(y^\tau)a<0,$$

$y(t,u^\tau)$ is decreasing in $t\geq 0$ by [3,Th.1]. An argument similar to the above one gives $\lim_{t\to\infty}y(t,u^\tau)=y^*$. This result together with $\lim_{t\to\infty}y(t,u^\varepsilon)=y^*$, (11) and Lemma 3 implies $\lim_{t\to\infty}y(t)=y^*$, as required.

The equation (1) may be written as $\dot{x}(t)=\bar{f}(x_t)$ if we define $\bar{f}: C\to\mathbf{R}^n$ by

$$\bar{f}(u)=\operatorname{diag}(u(0))\{a+\operatorname{diag}(b)u(0)+\int_{-r}^{0}[\mathrm{d}\mu(s)]u(s)\}. \tag{12}$$

Clearly, $\bar{f}$ is also completely continuous and $\bar{f}(u)\leq f(u)$ for all $u\in C_+$, where f is defined as in (9). Let $P=\int_{-r}^{0}\mathrm{d}\mu$, $Q=\operatorname{diag}(b)+P$. An argument similar to the proof of Theorem 1 shows that if $a>0$, $\sum_j a_{ij}<-b_i$ $(1\leq i\leq n)$, then Q^{-1} exists, but it is not necessarily $x^*=-Q^{-1}a>0$.

Theorem 2 Suppose $a>0$ and $\sum_j a_{ij}<-b_i$ $(1\leq i\leq n)$. If $x^*=-Q^{-1}a>0$, then for any $\varphi\in C_+^\circ$ we have $\lim_{t\to\infty}x(t,\varphi)=x^*$, unless

$$\inf_{t>0}\mathrm{d}(x(t,\varphi),\partial\mathbf{R}_+^n)=0. \tag{13}$$

Proof Fix $\varphi\in C_+^\circ$ and let $x(t)=x(t,\varphi)$. Suppose (13) fails, we prove $\lim_{t\to\infty}x(t)=x^*$. By Lemma 2, $x(t)>0$ on the whole domain. By Lamma 1, $x(t)\leq y(t,\varphi)$, hence $x(t)$ is bounded since $y(t,\varphi)$ is bounded by Theorem 1. By Lamma 3, $x(t)$ is defined on $[0,\infty)$. It suffices to prove that if $t_k\to\infty$ and $x(t_k)\to\bar{x}$, then $\bar{x}=x^*$. By Lemma 3, (1) has a bounded solution $y(t)$ on $\mathbf{R}$ with $y(0)=\bar{x}$. Note that $\bar{x}>0$, otherwise (13) follows. Thus $y(t)>0$ for all $t\in\mathbf{R}$ by Lemma 2. Let $z(t)=y(t)-x^*$. An immediate computation gives

$$\dot{z}_i(t)=y_i(t)[b_iz_i(t)+\sum_j\int_{-r}^{0}z_j(t+s)\,\mathrm{d}\mu_{ij}(s)],\quad i=1,2,\cdots,n. \tag{14}$$

Let $K=\sup_{t\leq 0}\|z(t)\|$, $\|\cdot\|$ denotes the max norm in $\mathbf{R}^n$. We claim that $\|z(t)\|\leq K$

for all $t\in\mathbf{R}$. Otherwise, $\|z(\tau)\|>K$ for some $\tau>0$, hence we may choose a smallest $\sigma\in(0,\tau)$ such that $|z_i(\sigma)|=\max_{0\le t\le\tau}\|z(t)\|$ for some i. We assume, e.g., $z_i(\sigma)>0$, and so $\|z(t)\|<z_i(\sigma)$ for all $t<\sigma$. But (14) implies

$$\frac{\dot z_i(\sigma)}{y_i(\sigma)}=b_iz_i(\sigma)+\sum_j\int_{-r}^{0}z_j(\sigma+s)\,\mathrm{d}\mu_{ij}(s)$$

$$\le b_iz_i(\sigma)+z_i(\sigma)\sum_j\int_{-r}^{0}\mathrm{d}v_{ij}=z_i(\sigma)\,(b_i+\sum_j a_{ij})<0,$$

this means $z_i(t)$ is strictly decreasing at $t=\sigma$, contradicts the choice of σ.

Replacing $z(t)$ by $z(\tau+t)$ we obtain

$$\|z(t)\|\le\sup_{s\le\tau}\|z(s)\| \tag{15}$$

for all $t,\tau\in\mathbf{R}$. Thus, to prove $\bar{x}=x^*$, i.e., $z(0)=0$, it suffices to show $\lim_{t\to-\infty}z(0)=0$. Let

$$p_i=\varliminf_{t\to-\infty}z_i(t),\ q_i=\varlimsup_{t\to-\infty}z_i(t),\quad p=\min_i p_i,\quad q=\max_i q_i.$$

We need only to prove $p=q=0$. If this is not valid, then there are at most three possible cases: (i) $q>0$, $q\ge -p$; (ii) $0<q<-p$; (iii) $q\le 0$, $p<0$. By considering $-z(t)$ instead of $z(t)$, the cases (ii) and (iii) can be reduced to the case (i), hence it suffices to consider the case (i). Suppose $q=q_i$, for given an $\varepsilon>0$ small enough, choose $\tau>0$ such that

$$|z_j(t)|<q+\varepsilon \text{ for all } t<-\tau \text{ and } j=1,2,\cdots,n. \tag{16}$$

If $\dot z_i(t)\ne 0$ for all sufficiently small t, then $z_i(t)$ monotonously converges to q as $t\to-\infty$, hence we may assume $|z_i(t)-q|<\varepsilon$ for all $t<-\tau$. This together with (16) implies

$$\frac{\dot z_i(t)}{y_i(t)}=b_iz_i(t)+\sum_j\int_{-r}^{0}z_j(t+s)\,\mathrm{d}\mu_{ij}(s)$$

$$\le b_i(q-\varepsilon)+(q+\varepsilon)\sum_j a_{ij}$$

$$=q(b_i+\sum_j a_{ij})+\varepsilon(\sum_j a_{ij}-b_i)=\beta_i<0$$

(Note that ε is small enough), it follows that

$$\ln\frac{y_i(-\tau)}{q+x_i^*}=\int_{-\infty}^{-\tau}\frac{\dot y_i(t)}{y_i(t)}\mathrm{d}t\le\int_{-\infty}^{-\tau}\beta_i\mathrm{d}t=-\infty,$$

a contradiction.

On the other hand, if there is a sequence $t_k\to-\infty$ such that $\dot z_i(t_k)=0$, then {$\{t_k\}$ can be chosen such that $z_i(t_k)\to q$ as $k\to\infty$. Thus, for sufficiently large k we have

$$0=\frac{\dot z_i(t_k)}{y_i(t_k)}=b_iz_i(t_k)+\sum_j\int_{-r}^{0}z_j(t_k+s)\,\mathrm{d}\mu_{ij}(s)$$

$$\le b_i(q-\varepsilon)+(q+\varepsilon)\sum_j a_{ij}<0,\ .$$

again a contradiction. Thus, the theorem is proved.

§ 4 Applications

The results of this paper can be applied to the differential difference equation

$$\dot{x}(t) = \mathrm{diag}(x(t))\,[a + \mathrm{diag}(b)\,x(t) + \sum_{k=0}^{N} A_k x(t - r_k)], \tag{17}$$

where $a, b \in \mathbf{R}^n$, $a > 0$, $A_k = (a_{ij}^k) \in \mathbf{R}^{n\times n}$, $a_{ii}^0 = 0$ ($i, j = 1, 2, \cdots, n$, $k = 0, 1, \cdots, N$), $0 = r_0 < r_1 < \cdots < r_N = r$. We may rewrite (17) in the standard form (1) by letting $\mu = \sum_{k=0}^{N} A_k \delta_k$, where δ_k denotes the Dirac measure at $s = -r_k$ for $k = 0, 1, \cdots, N$. Using the notations in §3, we have

$$\mu_{ij} = \sum_{k=0}^{N} a_{ij}^k \delta_k, \quad \nu_{ij} = |\mu_{ij}| = \sum_{k=0}^{N} |a_{ij}^k| \delta_k,$$

$$\int_{-r}^{0} \mathrm{d}\mu = \sum_{k=0}^{N} A_k, \quad Q = \mathrm{diag}(b) + \sum_{k=0}^{N} A_k.$$

Thus, we can apply Theorem 1 and Theorem 2 to obtain the following conclusion: If

$$\sum_{j=1}^{n} \sum_{k=0}^{N} |a_{ij}^k| < -b_i \quad \text{for } i = 1, 2, \cdots, n, \tag{18}$$

and $x^* = -(\mathrm{diag}(b) + \sum_{k=0}^{N} A_k)^{-1} a > 0$, then any positive solution $x(t)$ of (17) converges to x^* as $t \to \infty$ unless $\inf_{t>0} \mathrm{d}(x(t), \partial \mathbf{R}_+^n) = 0$, if, in addition, $A_k \geq 0$ for $k = 0, 1, \cdots, N$, N, then any positive solution of (17) necessarily converges to x^* as $t \to \infty$.

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一类积微分方程的全局渐近状态

胡适耕

(华中理工大学数学系，武汉)

摘要

本文考虑形如 $\dot{x}(t) = \mathrm{diag}(x(t))\{a + \mathrm{diag}(b)x(t) + \int_{-r}^{0}[\mathrm{d}\mu(s)]x(t+s)\}$ 的微积分方程. 在对 μ 的一定假设下，我们证明了：所述方程的任何正解或者渐近于同一平衡状态，或者能任意接近正卦限 $\mathbf{R}_+^n$ 的边界.