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A Minimax Principle without Differentiability*

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Abstract

By using the theory concerning with quasi-tangency stated by D. Motreanu and N. H. Pavel, and depeloped by author, we astablish a minimax principle for locally Lipschitzian functionals on locally convex closed subsets of Banach spaces.

Introduction

The theory concerning with quasi-tangency stated by D. Motreanu and Parel [5,6], developed by author [11] extends our research areas from manifolds to subsets of manifolds, where a full manifold structure is not necessary. On the other hand Chang[4] has generalized some well known critical point theorems to locally Lipschitzian functionals on Banach spaces. The purpose of the present paper is to establish a minimax principle for locally Lipschitzian functionals on locally convex closed subsets of a Banach space.

The paper is organized as follows. We will recall briefly the general theory of relative tangency first. Then we will prove the existence theorem of pseudo-gradient vector field for a locally Lipschitzian functional on a locally convex closed subset of a Banach space. As natural corollaries of the existence theorem we may give a existence theorem of the minimization, a strong deformation theorem, a mountain-pass theorem and a Lusternik-Schnirelman theorem, in this satution.

Through the paper we denote X to be a Banach space with a norm $\|\cdot\|$, S to be a nonempty subset of X, and f to be a locally Lipschitzian functional defined on an open neighborhood U of S.

Definition 1. A vector $v\in X$ is called tangent to the subset S of X at $x\in S$ if

$$\lim_{h\to 0}\frac{1}{h}d(x+hv,\ S)=0 \text{ with } h\in R, \tag{1}$$

where $d(.,.)$ is the distance induced by the norm on X, i.e.,

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$$d(u, A) = \inf_{a \in A} \|u - a\|.$$

The following proposition proved in [11] is basic for funther development.

Proposition 1 The following statements are ture:

(i) the set of tangent vectors to S at $x \in S$ is a cone,

(ii) if the set S is locally convex then the set of tangent vectors to S at $x \in S$ is a vector subspace of X,

(iii) if S is locally convex, and if W is any convex subset of S then the set of tangent vectors to S at point $\lambda_1 x + (1-\lambda_1) y$ is the same as the set of tangent vectors to S at point $\lambda_2 x + (1-\lambda_2) y$ for all $\lambda_1, \lambda_2 \in (0, 1)$, where $I \equiv \{\lambda x + (1-\lambda) y;\ \lambda \in [0, 1]\} \subset W$. Furthermore, if $\{\lambda x + (1-\lambda) y;\ \lambda \in [0, 1]\} \subset W$, and if v is tangent to S at y then v is also tangent to S at any point $\lambda x + (1-\lambda) y$, $\lambda \in (0, 1)$.

It is well known that for any locally Lipschitzian functional f defined on an open neighborhood U of S in X we may define a generalized derivative of f at any point $x \in U$, which is denoted by $\partial f(x)$.

$\partial f(x)$ is W^*-compact subset of the dual space X^* of X. We define $\lambda_S(x) = \inf_{w \in \partial f(x)} \{\|w\|_{*,S}\}$, where $\|w\|_{*,S} = \sup\{\langle w, v\rangle;\ \|v\| = 1,\ v \text{ is tangent to } S \text{ at } x\}\}$.

Definition 2 A point $x \in S$ is called a generalized critical point of f relative to S provided the equation $\lambda_S(x) = 0$.

We may also introduce a compactness condition as follows.

Definition 3 f is satisfied Condition(C) of Palais and Smale relative to S if for and sequence $\{x_n\}$ in S with bounded $\{f(x_n)\}$ and $(\lambda_S, x_n) \to 0$, then there is a convergent subsequence of $\{x_n\}$.

We use the following usual notations.

$$A_c = \{x \in S;\ f(x) \le c\}.\quad K_c = \{x \in S;\ 0 = \lambda_S(x),\ f(x) = c\},$$
$$B(c, \varepsilon, \delta) = \overline{A_{c+\varepsilon}} - \overline{A_{c-\varepsilon}} - \overline{N_\delta(K_c)},$$

where $N_\delta(K_c)$ is the δ-neighborhood of K_c.

It is easy to see that if f satisfies Condition (C) of Palais and Smale relative to S, then K_c is a compact subset of X.

The following lemma is immediately obtained.

Lemma 1 Suppose that f satisfies Condition (C) of Palais and Smale relative to S, then for each $\delta > 0$ there exist η, $\varepsilon > 0$ such that

$$\lambda_S(x) \geqslant \eta \text{ for all } x \in B(c, \varepsilon, \delta).$$

Lemma 2 For each $x \in S$ there exists a $w_0 \in \partial f(x)$ such that $\|w_0\|_{*,S} = \lambda_S(x)$.

Proof It is easy to see that the map from $\partial f(x)$ to $\mathbf{R}$ defined by $w \mapsto \|w\|_{*,S}$ is a weak*-lower semicontinuous mapping. Indeed, we assume that it is not ture.

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Then there esists a weak*-convergent sequence $\{w_n\}$ in $\partial f(x)$ with weak* limit w_0 such that

$\lim\limits_{n\to\infty}\|w_n\|_{*,S}<\|w_0\|_{*,S}$. Denote $\delta=\|w_0\|_{*,S}-\lim\limits_{n\to\infty}\|w_n\|_{*,S}$.

By the definition of the norm $\|\cdot\|_{*,S}$ we may pich a point $v\in X$ such that v is tangent to S at x with $\|v\|=1$ and

$$w_0(v)\geqslant\|w_0\|_{*,S}-\frac{\delta}{3}. \tag{2}$$

For each sufficient large n we have

$$w_n(v)\leqslant\|w_n\|_{*,S}\leqslant\|w_0\|_{*,S}-\frac{2\delta}{3}. \tag{3}$$

Combining (2) and (3) we obtain an inequality $w_n(v)\leqslant w_0(v)-\frac{\delta}{3}$. This contradicts to the weak*-convergency of $\{w_n\}$.

Let $\{w_n\}\subset\partial f(x)$ such that $\{\|w_n\|_{*,S}\}$ is a monotonically decreasing sequence of numbers with limit $\lambda_S(x)=\inf\limits_{w\in\partial f(x)}\|w\|_{*,S}$. Since $\partial f(x)$ is a non-empty weak*-compact convex subset of X^*, we may assume that $\{w_n\}$ is weak*-convergent to $w_0\in\partial f(x)$. By the lower semi-continuity of $w\mapsto\|w\|_{*,S}$, we have

$$\lambda_S(x)=\lim_{n\to\infty}\|w_n\|_{*,S}\geqslant\lambda_S(x).$$

Thus $\|w_0\|_{*,S}=\lambda_S(x)$.

Now we can prove the following lemma which is concerning with the existence of a pseudo-gradient vector field.

Lemma 3 Suppose that f satisfies Condition (C) of Palais and Smale relative to S, and S is locally convex closed subset of X. Then for each closed subset S_1 of $S^{\#}=\{x\in S;\ \lambda_S(x)\neq 0\}$ there exists a locally Lipschitzian vector field V on S_1 such that for each $x\in S_1$, $V(x)$ is tangent to S at x, $\|V(x)\|\leqslant 1$ and $\langle w, V(x)\rangle\geqslant\frac{\eta}{2}$ for each $w\in\partial f(x)$, where $\eta=\inf\{\lambda_S(x);\ x\in S_1\}$.

Proof First of all, we may see that $\eta>0$ because of Condition (C) of Palais and Smale relative to S. For $x_0\in S_1$ there exists a $w_0\in\partial f(x_0)$ such that $\|w_0\|_{*,S}=\lambda_S(x_0)$, by Lemma 2. Since $\partial f(x_0)$ is a weak*-compact convex subset of X^*, and since for any $0<r<\|w_0\|_{*,S}$ $\bar{B}(0,r)\cap\partial f(x_0)=\varphi$, where $\bar{B}(0,r)$ denotes the weak*-closure of the r-ballin X^* with center 0, there exists a $h\in(X^*_{W^*})^*$ such that $\langle h, w\rangle>\langle h, x^*\rangle$, for all $w\in\partial f(x_0)$ and $\|x^*\|\leq r$ by Hahn-Banach theorem. On the other hand, we can find a $v\in X$, such that

$\langle h, x^*\rangle=\langle x^*, v\rangle$, for each $x^*\in X^*$ (see p. 155(12)).

Then we have $\langle w,v\rangle>\langle x^*,v\rangle$, for each $w\in\partial f(x_0)$ and $\|x^*\|\leqslant r$. It follows that

$$\inf_{w\in\partial f(x_0)}\langle w, v\rangle>r\|v\|.$$

Let $r=\frac{\eta}{2}$ anh $\|v\|=1$. By the weak*-upper semicontinuity of $\partial f(x)$, and by using proposition 1 we can find a convex neighborhood $N(x_0)$ of x_0 such that $\inf\limits_{w\in\partial f(x)}\langle w, v\rangle$

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$>\frac{\eta}{2}$, for all $x\in N(x_0)\cap S_1$ and v is tangent to S at x.

Let $\{\beta_a(x)\}_{a\in\mathscr{A}}$ be a partition of unity of a locally finite refirement of the open covering $\{N(x)\cap S_1\}$ of S_1 and let $v(x)=\sum_{a\in\mathscr{A}}\beta_a(x)v_a$. Then V is required.

Lemma 4 Suppose that S is a locally convex closed subset of X, and that f satisfies Condition (C) of Palais and Smale relative to S. Let $c\in\mathbf{R}$, $\bar{\varepsilon},\delta$ and $\eta\geq 0$ such that $\lambda_S(x)\geq\eta$ for all $x\in B(c,\bar{\varepsilon},\delta)$. For each $\varepsilon\in(0,\bar{\varepsilon})$ we define a vector field on S as $W_\varepsilon(x)=g(x)\cdot\bar{g}(x)\cdot V_\varepsilon(x)$ where g and $\bar{g}$ are continuous functions defined on S with range $[0,1]$ and satisfy

$$g(x)=\begin{cases}1, & x\in A_{c+\varepsilon}-A_{c-\varepsilon}\\ 0, & x\notin A_{c+\varepsilon}-A_{c-\varepsilon}\end{cases},$$

$$\bar{g}(x)=\begin{cases}1, & x\notin N_{4\delta}(K_c)\\ 0, & x\in N_{2\delta}(K_c)\end{cases}, \text{ respectively}$$

and the vector field V_ε is constructed in $\overline{B(c,\varepsilon,\delta)}$ as in Lemma 3. Such a defined W_ε may be seen as a vector field on whole S.

Let $\varphi(x,t)$ be the flow of the following Cauchy problam:

$$\frac{d}{dx}\varphi(x,t)=-W_\varepsilon(\varphi(x,t)),\quad \varphi(x,0)=x.$$

Then the following statements hold:

(i) the function f along $\varphi(x,t)$ is non-increasing on t,

(ii) $\|\varphi(x,t)-x\|\leq t$,

(iii) $f(x)-f(\varphi(x,t))\geq(\frac{\eta}{2})t$, for $\varphi(x,t)\in B(c,\varepsilon,4\delta)$ and $s\in[0,t)$.

Proof

$$\|\varphi(x,t)-x\|=\|\int_0^t\frac{d}{ds}\varphi(x,s)ds\|\leq\int_0^t\|W_\varepsilon(\varphi(x,s))\|ds\leq\int_0^t ds=t.$$

Let $h(t)=f(\varphi(x,t))$. The following calculation

$$\begin{aligned}h'(t)&\leq\max\{\langle w,\frac{d}{ds}\varphi(x,s)\rangle;\ w\in\partial f(\varphi(x,s))\}\quad \text{a.e.}\\ &=-\min\{\langle w,W_\varepsilon(\varphi(x,s))\rangle;\ w\in\partial f(\varphi(x,s))\}\\ &\leq\begin{cases}-\eta/2, & \varphi(x,s)\in B(c,\varepsilon,4\delta)\\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

shows that the function f along $\varphi(x,t)$ is non-increasing and

$$f(x)-f(\varphi(x,t))=-\int_0^t h'(s)\,ds>\frac{\eta}{2}t,\ \text{if } \varphi(x,t)\in B(c,\varepsilon,4\delta),\ s\in[0,t).$$

The following theorem is the key step in developing the minimax principle.

Theorem 1 (Deformation Lemma) Suppose that S is a locally convex closed subset of X, and that f satisfies Condition (C) of Palais and Smale ralative to S. If c is a real number and N is any neighborhood of K_c then for any $\varepsilon_0>0$ there exists $\varepsilon\in(0,\varepsilon_0)$ and a homomorphism $\psi: S\to S$ such that

(i) $\psi(x)=x$ for $x\notin A_{c+\varepsilon_0}-A_{c-\varepsilon_0}$,

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(ii) $\psi(A_{c+\varepsilon}/N) \subset A_{c-\varepsilon}$,

(iii) if $K_c = \varphi$, then $\psi(A_{c+\varepsilon}) \subset A_{c-\varepsilon}$.

Proof We choose $\delta > 0$ sush that $N_{6\delta}(K_c) \subset N$. Let η and $\bar{\varepsilon}$ be the numbers determined in Lemma 1. Without loss of generality we may assume that $\varepsilon <$ $\min(\varepsilon_0, \eta\delta/4)$ and take $\varepsilon \in (0, \bar{\varepsilon})$. For the flow $\varphi(x, t)$ generated by W_t in Lemma 4 we have

$$\varphi(x, t) = x \text{ for all } x \notin A_{c+\varepsilon_0} - A_{c-\varepsilon_0} \text{ and all } t.$$

We now observe those trajectories $\varphi(x,t)$ emanating from x in $A_{c+\varepsilon} \setminus N_{6\delta}(K_c)$. Let $t_0 = (\frac{4}{\eta})\bar{\varepsilon}$. If $x \in A_{c-\varepsilon}$, it is obvious that $\varphi(x, t_0) = x \in A_{c-\varepsilon}$, so we may restrict x only on the subset $B(c, \varepsilon, 6\delta)$.

We expect to prove $\varphi(x, t_0) \in A_{c-\varepsilon}$. If not, then $\varphi(x, s) \in A_{c+\varepsilon} - A_{c-\varepsilon}$ for all $s \in [0, t_0]$. The inequalities $\|\varphi(x, s) - x\| \leq s \leq \frac{2}{\eta}(f(x) - f(\varphi(x, s))) < \frac{2}{\eta} \cdot 2\bar{\varepsilon} < \delta$, imply that if $x \notin N_{6\delta}(K_c)$ then $\varphi(x, s) \notin N_{4\delta}(K_c)$ for each $s \in [0, t_0]$. Then $\varphi(x, s) \in B(c, \varepsilon, 4\delta)$ for all $s \in [0, t_0]$. By Lemma 4 again we have $f(x) - f(\varphi(x, t_0)) \geq (\frac{\eta}{2}) \cdot t_0 = 2\bar{\varepsilon} > 2\varepsilon$. It contracts to the fact that $x \in A_{c+\varepsilon} - A_{c-\varepsilon} - N_{6\delta}(K_c)$ and $\varphi(x, t_0) \in A_{c+\varepsilon} - A_{c-\varepsilon} - N_{4\delta}(K_c)$. Taking $\psi(x) = \varphi(x, t_0)$ we find ψ satisfies all requirements in the theorem.

Remark The proof of the above theorem is along the line of the proof given by Chang [4].

The minimax principle, which is a series of existence theorems of critical points based on, is derived by the deformation lemma. A lot of theorems, such as minimization theorem, Lusternik-Schnirelman category theory and mountain-pass type theorems hold ture for locally Lipschitzian functionals on locally convex closed subsets of Banach spaces. We only list some of them and omit the detail proofs.

Theorem 2 (minimization) Suppose that S is a locally convex closed subset of X, and that f satisfies Condition (C) of Palais and Smale relative to S and is bounded from below on S. Then $f|_S$ attains its greatest lower bound.

Theorem 3 (Lusternik-Schnirelman) Suppose that f satisfies Condition (C) of Palais and Smale relative to S, where S is a locally convex closed subset of X. If $-\infty < c = c_{m+1} = c_{m+2} = \cdots = c_{m+k} < \infty$, then f has at lcast k distinct critical points relative to S in the level $f^{-1}(c)$, where $c_i = \inf\limits_{A \in \mathscr{F}_i} \sup\limits_{x \in A} f(x)$, $\mathscr{F}_i = \{A \subset S$, $\operatorname{cat}(A, X) \geq i\}$, $\operatorname{cat}(A, X)$ denotes the Lusternik-Schnirelman category of A in X.

Corollary Under the assumptions of Theorem 3, f has at least $\operatorname{cat}(S, X)$ distinct critical points relative to S. ■

Theorem 4 (Rabinowitz) Let S be a locally convex closed subset of X, which contains the origin of X. Suppose that f satisfies Condition (C) of Palais

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and Smale relative to S. If there exists a decomposition of $X=X_1\oplus X_2$ with a finite-dimensional X_1 and if there exist constants b_1 and b_2 with $b_1<b_2$ and a neighborhood N of X_1, such that $f|_{X_2\cap S}\geq b_2$, $f|_{\partial N\cap S}\leq b_1$; then f has a critical point relative to S.

All of results in [1], [2] and [7] can be extended in the same way.

References

[1] H. Amann, Math. Z. 196 (1979), 127—166.

[2] A. Ambrosetti and P. H. Rabinowitz, J. Functional Analysis, 14 (1973), 349—381.

[3] V. Benci and P. H. Rabinowitz, Invent. Math., 52 (1979), 241—273.

[4] K. C. Chang, J. Math. Anal. Appl., 80 (1981) 102—129.

[5] D. Motreanu and N. H. Pavel, J. Math. Anal. Appl. 88 (1982) 116—132.

[6] I. Motreanu, J. Math. Anal. Appl., 117 (1986), 128—137.

[7] W. M. Ni, J. Analyse Math., 37 (1980), 248—275.

[8] R. S. Palais, Topology, 2 (1963), 299—340.

[9] R. S. Palais, Topology, 5 (1966), 115—132.

[10] P. H. Rabinowitz, CBMS Regional Conf. Ser. in Math., No. 65, Amer. Math. Soc., Providence, R. I., 1986.

[11] T. Wang, A minimax principle on closed subsets of Banach manifolds and its applications, to appear.

[12] J. L. Kelley and I. Namioka, Linear Topological Spaces, Van Norstrand (1963).

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