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Some Algebraic Relations Between p-models*

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Abstract

We define the partial models as the models of 3-valued logic. The Kleene semantics for the first-order language is given. Some algebraic relations between p-models, such as isomorphism, homomorphism and extension, are discussed. Using the method of diagram, we give another description of these relations. The fixed point theorem is proved for certain operators on p-models.

We shall study some algebraic relations between the partial models which are defined as the models of three-valued logic. About the syntax of the first-order language and some notations in model theory, reades can refer to [1] and [3].

We first define the partial model. Let L be a language, we write $L=\{C_i\}_{i\in I}\cup\{R_j\}_{j\in J}$, where $C_i's$ are constants and $R_j's$ are predicates.

Definition 1 A partial model (p-model) for L is a structure $\mathfrak{A}=\langle A, \{C_i^{\mathfrak{A}}\}_{i\in I}, \{R_j^{\mathfrak{A}}\}_{j\in J}\rangle$ such that

1) $A\neq\varnothing$ and A is denoted by $|\mathfrak{A}|$.

2) For all $i\in I$, $C_i^{\mathfrak{A}}\in A$.

3) For all $j\in J$, $R_j^{\mathfrak{A}}$ is a partial function from A^k to $\{0,1\}$ if R_j is K-ary.

Some rolations between p-models define as follows:

Definition 2 Let $\mathfrak{A}, \mathfrak{B}$ be two p-models for L, we write $\mathfrak{A}=\langle A, \{C_i^{\mathfrak{A}}\}_{i\in I}, \{R_j^{\mathfrak{A}}\}_{j\in J}\rangle$, $\mathfrak{A}=\langle B, \{C_i^{\mathfrak{B}}\}_{i\in I}, \{R_j^{\mathfrak{B}}\}_{j\in J}\rangle$.

2.1) We say that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ ($\mathfrak{A}\subseteq\mathfrak{B}$) iff (1) $A\subseteq B$, (2) for all $i\in I$, $C_i^{\mathfrak{A}}=C_i^{\mathfrak{B}}$, (3) for all $j\in J$, if R_j is K-ary, then for all $\bar{a}\in A^k$, $R_j^{\mathfrak{A}}(\bar{a})=R_j^{\mathfrak{B}}(a)$.

2.2) We say that $\mathfrak{A}$ is homomorphic to $\mathfrak{B}$ ($\mathfrak{A}\simeq\mathfrak{B}$) iff there is a function f mapping A onto B such that (1) for all $i\in I$, $f(C_i^{\mathfrak{A}})=C_i^{\mathfrak{B}}$, (2) for all $j\in J$, if R_j is K-ary, then $R_j^{\mathfrak{A}}(\bar{a})=R_j^{\mathfrak{B}}(f(\bar{a}))$ for all $\bar{a}\in A^k$.

2.3) We say that $\mathfrak{A}$ is isomophic to $\mathfrak{B}$ ($\mathfrak{A}\cong\mathfrak{B}$) iff there is a function f such that f is 1-1 and f is a homomorphic mapping from A onto B.

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2.4) We say that $\mathfrak{B}$ is a extension of $\mathfrak{A}$ ($\mathfrak{A} \leq \mathfrak{B}$) iff (1) $|\mathfrak{A}| = |\mathfrak{B}|$, (2) for all $i \in I$, $C_i^{\mathfrak{A}} = C_i^{\mathfrak{B}}$, (3) for all $j \in J$, if R_j is K-ary, then for all $\bar{a} \in A^k$, $R_j^{\mathfrak{A}}(\bar{a}) = 1 \Rightarrow R_j^{\mathfrak{B}}(\bar{a}) = 1$ and $R_j^{\mathfrak{A}}(\bar{a}) = 0 \Rightarrow R_j^{\mathfrak{B}}(\bar{a}) = 0$.

Defination 3 Suppose that $L \subset L'$, $\mathfrak{A}$ is a p-model for L and $\mathfrak{A}'$ is a p-model for L'. We say that $\mathfrak{A}'$ is an expansion of $\mathfrak{A}$ iff $|\mathfrak{A}| = |\mathfrak{A}'|$ and for all $s \in L$, $s^{\mathfrak{A}} = s^{\mathfrak{A}'}$.

Using the Kleene semantics, we have the following:

Definition 4 Let φ be an expression of L, $\mathfrak{A}$ be a p-model of L and $z \in {}^V|\mathfrak{A}|$, where $V = \{x_0, x_1, \cdots\}$ which is the set of all variables of L, we define the value $\varphi^{\mathfrak{A}}\langle z\rangle$ by induction on the length of φ:

(1) $\varphi \equiv x \in V$, $\varphi^{\mathfrak{A}}\langle z\rangle = z(x)$

(2) $\varphi \equiv C_i$, $\varphi^{\mathfrak{A}}\langle z\rangle = C_i^{\mathfrak{A}}$

(3) $\varphi \equiv t_1 = t_2$, $\varphi^{\mathfrak{A}}\langle z\rangle = \begin{cases} 1 & t_1^{\mathfrak{A}}\langle z\rangle = t_2^{\mathfrak{A}}\langle z\rangle \\ 0 & t_1^{\mathfrak{A}}\langle z\rangle \neq t_2^{\mathfrak{A}}\langle z\rangle \end{cases}$

(4) $\varphi \equiv R_j(t_1, \cdots, t_k)$, $\varphi^{\mathfrak{A}}\langle z\rangle = \begin{cases} 1 & \text{if } R_j^{\mathfrak{A}}(t_1^{\mathfrak{A}}\langle z\rangle, \cdots, t_k^{\mathfrak{A}}\langle z\rangle) = 1 \\ 0 & \text{if } R_j^{\mathfrak{A}}(t_1^{\mathfrak{A}}\langle z\rangle, \cdots, t_k^{\mathfrak{A}}\langle z\rangle) = 0 \\ u & \text{o. w.} \end{cases}$

(5) $\varphi \equiv \neg\psi$, $\varphi^{\mathfrak{A}}\langle z\rangle = 1$ if $\psi^{\mathfrak{A}}\langle z\rangle = 0$, $\varphi^{\mathfrak{A}}\langle z\rangle = 0$, if $\psi^{\mathfrak{A}}\langle z\rangle = 1$ and $\varphi^{\mathfrak{A}}\langle z\rangle = u$ if $\psi^{\mathfrak{A}}\langle z\rangle = u$

(6) $\varphi \equiv \psi_1 \wedge \psi_2$, $\varphi^{\mathfrak{A}}\langle z\rangle = \begin{cases} 1 & \text{if } \psi_1^{\mathfrak{A}}\langle z\rangle = \psi_2^{\mathfrak{A}}\langle z\rangle = 1 \\ 0 & \text{if } \psi_1^{\mathfrak{A}}\langle z\rangle = 0 \text{ or } \psi_2^{\mathfrak{A}}\langle z\rangle = 0 \\ u & \text{o. w.} \end{cases}$

(7) $\varphi \equiv \exists x\psi$, $\varphi^{\mathfrak{A}}\langle z\rangle = \begin{cases} 1 & \text{if for some } a \in A,\ \psi^{\mathfrak{A}}\langle z(\frac{x}{a})\rangle = 1 \\ 0 & \text{if for all } a \in A,\ \psi^{\mathfrak{A}}\langle z(\frac{x}{a})\rangle = 0 \\ u & \text{o. w.} \end{cases}$

In this definition, we adopt Kleene's strong three-valued connectives. (See [2]).

Theorem 5 If $\mathfrak{A} \cong \mathfrak{B}$, then $\varphi^{\mathfrak{A}} = \varphi^{\mathfrak{B}}$ for any sentence φ.

Proof Let g be an isomophism from $\mathfrak{A}$ to $\mathfrak{B}$, we prove by induction on formulas that for all $z \in {}^V|\mathfrak{A}|$ and all formulas φ, $\varphi^{\mathfrak{A}}\langle z\rangle = \varphi^{\mathfrak{B}}\langle g \circ z\rangle$. We consider only the case φ has the form $\exists x\psi$. Let φ be $\exists x\psi$, $\varphi^{\mathfrak{A}}\langle z\rangle = 1$ iff for some $a \in |\mathfrak{A}|$, $\psi^{\mathfrak{A}}\langle z(\frac{x}{a})\rangle = 1$ iff for some $a \in |\mathfrak{A}|$, $\psi^{\mathfrak{B}}\langle g \circ z(\frac{x}{a})\rangle = 1$ (by I.H.) iff for some $a \in |\mathfrak{A}|$, $\psi^{\mathfrak{B}}\langle (g \circ z)(\frac{x}{g(a)})\rangle = 1$ iff $\varphi^{\mathfrak{B}}\langle g \circ z\rangle = 1$. In the same way, $\varphi^{\mathfrak{A}}\langle z\rangle = 0$ iff $\varphi^{\mathfrak{B}}\langle g \circ z\rangle = 0$, whence $\varphi^{\mathfrak{A}}\langle z\rangle = u$ iff $\varphi^{\mathfrak{B}}\langle g \circ z\rangle = u$, hence $\varphi^{\mathfrak{A}}\langle z\rangle = \varphi^{\mathfrak{B}}\langle g \circ z\rangle$. When φ is a sentence, we have $\varphi^{\mathfrak{A}} = \varphi^{\mathfrak{B}}$.

Theorem 6 If $\mathfrak{A} \leq \mathfrak{B}$, then $\varphi^{\mathfrak{A}}\langle z\rangle = 1 \Rightarrow \varphi^{\mathfrak{B}}\langle z\rangle = 1$ and $\varphi^{\mathfrak{A}}\langle z\rangle = 0 \Rightarrow \varphi^{\mathfrak{B}}\langle z\rangle = 0$.

Proof By induction on φ, we prove $\varphi^{\mathfrak{A}}\langle z\rangle = (\begin{smallmatrix}1\\0\end{smallmatrix}) \Rightarrow \varphi^{\mathfrak{B}}\langle z\rangle = (\begin{smallmatrix}1\\0\end{smallmatrix})$.

Case 1: $\varphi \equiv t_1 = t_2$, $\varphi^{\mathfrak{A}}\langle z\rangle = 1 \Rightarrow t_1^{\mathfrak{A}}\langle z\rangle = t_2^{\mathfrak{A}}\langle z\rangle \Rightarrow t_1^{\mathfrak{B}}\langle z\rangle = t_2^{\mathfrak{B}}\langle z\rangle \Rightarrow \varphi^{\mathfrak{B}}\langle z\rangle = 1$. In the same way, $\varphi^{\mathfrak{A}}\langle z\rangle=0\Rightarrow\varphi^{\mathfrak{B}}\langle z\rangle=0$.

Case 2 $\varphi\equiv R_j(t_1,\cdots,t_k)$, $\varphi^{\mathfrak{A}}\langle z\rangle=1\Rightarrow R_j^{\mathfrak{A}}(t_1^{\mathfrak{A}}\langle z\rangle,\cdots,t_k^{\mathfrak{A}}\langle z\rangle)=1\Rightarrow R_j^{\mathfrak{B}}(t_1^{\mathfrak{A}}\langle z\rangle,\cdots,t_k^{\mathfrak{A}}\langle z\rangle)=1\Rightarrow R_j^{\mathfrak{B}}(t_1^{\mathfrak{B}}\langle z\rangle,\cdots,t_k^{\mathfrak{B}}\langle z\rangle)=1\Rightarrow\varphi^{\mathfrak{B}}\langle z\rangle=1$. In the same way, $\varphi^{\mathfrak{A}}\langle z\rangle=0\Rightarrow\varphi^{\mathfrak{B}}\langle z\rangle=0$.

Case 3 $\varphi\equiv\neg\psi$, $\varphi^{\mathfrak{A}}\langle z\rangle=1\Rightarrow\psi^{\mathfrak{A}}\langle z\rangle=0\Rightarrow\psi^{\mathfrak{B}}\langle z\rangle=0\Rightarrow\varphi^{\mathfrak{B}}\langle z\rangle=1$. In the same way, $\varphi^{\mathfrak{A}}\langle z\rangle=0\Rightarrow\varphi^{\mathfrak{B}}\langle z\rangle=0$.

Case 4 $\varphi\equiv\psi_1\wedge\psi_2$, $\varphi^{\mathfrak{A}}\langle z\rangle=1\Rightarrow\psi_i^{\mathfrak{A}}\langle z\rangle=1$ $(i=1,2)\Rightarrow\psi_i^{\mathfrak{B}}\langle z\rangle=1$ (by I. H.)$\Rightarrow\varphi^{\mathfrak{B}}\langle z\rangle=1$.

Case 5 $\varphi=\exists x\psi$, $\varphi^{\mathfrak{A}}\langle z\rangle=1\Rightarrow$for some $a\in A$, $\psi^{\mathfrak{A}}\langle z(\frac{x}{a})\rangle=1\Rightarrow$ for some $a\in A$, $\psi^{\mathfrak{B}}\langle z(\frac{x}{a})\rangle=1\Rightarrow\varphi^{\mathfrak{A}}\langle z\rangle=1$ (since $A=B$). In the same way, $\varphi^{\mathfrak{A}}\langle z\rangle=0\Rightarrow\varphi^{\mathfrak{A}}\langle z\rangle=0$.

Definition 7 Let $\mathfrak{A}$ be a p-model for L, $L_A=L\cup\{C_a: a\in A\}$, we expand $\mathfrak{A}$ to $\mathfrak{A}_A=(\mathfrak{A},C_a^{\mathfrak{A}_A})$ $a\in A$ where $A=|\mathfrak{A}|$ and $C_a^{\mathfrak{A}_A}=a$, so $\mathfrak{A}_A=(\mathfrak{A},a)$ $a\in A$. We define the positive diagram of $\mathfrak{A}$ ($P\mathfrak{A}$), the negative diagram of $\mathfrak{A}$ ($N\mathfrak{A}$) and the diagram of $\mathfrak{A}$ ($\Delta\mathfrak{A}$) as the following: $P\mathfrak{A}=\{\sigma: \sigma$ is an atomic sentence of L_A and $\sigma^{\mathfrak{A}_A}=1\}$. $N\mathfrak{A}=\{\neg\sigma: \sigma$ is an atomic sentence of L_A and $\sigma^{\mathfrak{A}_A}=0\}$. $\Delta\mathfrak{A}=P\mathfrak{A}\cup N\mathfrak{A}$.

Theorem 8 Let $\mathfrak{A}$, $\mathfrak{B}$ be p-models for L, $\mathfrak{A}$ is homomorphic to a submodel of $\mathfrak{B}$ iff some expansion of $\mathfrak{B}$ is a model of $P\mathfrak{A}$.

Proof "$\Rightarrow$", Suppose that $\mathfrak{A}\simeq g\,\mathfrak{B}'\subseteq\mathfrak{B}$, we expand $\mathfrak{B}$ to $\mathfrak{B}^+=(\mathfrak{B},C_a^{\mathfrak{B}^+})a\in A$, $\mathfrak{B}^+$ is a p-model for L_A and $C_a^{\mathfrak{B}^+}=g(a)$ $(a\in A)$, so $\mathfrak{B}^+=(\mathfrak{B},g(a))\,a\in A$. since g is a homomorphism, $\sigma^{\mathfrak{A}_A}=1\Rightarrow\sigma^{\mathfrak{B}^+}=1$ for any atomic sentence σ, hence $\mathfrak{B}^+$ is a model of $P\mathfrak{A}$.

"$\Leftarrow$". Suppose that some expansion of $\mathfrak{B}$, say $\mathfrak{B}^+$, is a model of $P\mathfrak{A}$, then for all $a\in A$, $C_a^{\mathfrak{B}^+}\in B$, so we define a mapping g from A to B: $g(a)=C_a^{\mathfrak{B}^+}$. It is easy to verify that g is a homomorphic mapping. Since $R_j^{\mathfrak{A}}(a_1,\cdots,a_k)=1\Rightarrow R_j^{\mathfrak{A}_A}\cdot(C_{a_1}^{\mathfrak{A}_A},\cdots,C_{a_k}^{\mathfrak{A}_A})=1\Rightarrow(R_j(C_{a_1},\cdots,C_{a_k}))^{\mathfrak{A}_A}=1\Rightarrow R_j(C_{a_1},\cdots,C_{a_k})\in P\mathfrak{A}\Rightarrow(R_j(C_{a_1},\cdots,C_{a_k}))^{\mathfrak{B}^+}=1\Rightarrow R_j^{\mathfrak{B}}(g(a_1),\cdots,g(a_k))=1$, g is homomorphic.

Theorem 9 $\mathfrak{A}$ is isomorphic to a submodel of $\mathfrak{B}$ iff some expansion of $\mathfrak{B}$ is a model of $\Delta\mathfrak{A}$.

The proof of this theorem is similar to that of th. 8.

Theorem 10 $\mathfrak{A}\leq\mathfrak{B}$ iff $\mathfrak{B}_A$ is a model of $\Delta\mathfrak{A}$ where $\mathfrak{B}_A=(\mathfrak{B},a)a\in A$.

Proof "$\Rightarrow$". Assume $\mathfrak{A}\leq\mathfrak{B}$, then $\mathfrak{A}_A\leq\mathfrak{B}_A$. By Thm 6, we have $\sigma^{\mathfrak{A}_A}=\binom{1}{0}\Rightarrow\sigma^{\mathfrak{B}_A}=\binom{1}{0}$ hence $\mathfrak{B}_A$ is a model of $\Delta\mathfrak{A}$.

"$\Leftarrow$". Assume $\mathfrak{B}_A$ is a model of $\Delta\mathfrak{A}$, $\forall i\in I$, If $C_i^{\mathfrak{A}}=a$, then $(C_i=C_a)^{\mathfrak{A}}=1$, whence $(C_i=C_a)\in\Delta\mathfrak{A}$ hence $(C_i=C_a)^{\mathfrak{B}}=1$ i.e., $C_i^{\mathfrak{B}}=a$. We have $C_i^{\mathfrak{A}}=C_i^{\mathfrak{B}}$. In addition, if $R_j^{\mathfrak{A}}(a_1,\cdots,a_k)=1$, then $R_j(C_{a_1},\cdots,C_{a_k})\in\Delta\mathfrak{A}$, so $(R_j(C_{a_1},\cdots,C_{a_k}))^{\mathfrak{B}_A}=1$ whence $R_j^{\mathfrak{B}}(a_1,\cdots,a_k)=1$. In the same way, we have if $R_j^{\mathfrak{A}}(a_1,\cdots,a_k)=0$, then $R_i^{\mathfrak{B}}(a_1,\cdots,a_k)=0$. Whence $R_j^{\mathfrak{A}}(\bar{a})=R_j^{\mathfrak{B}}(\bar{a})$ hence $\mathfrak{A}\leq\mathfrak{B}$.

In artificial intelligence, we need to discuss operators on p-models. Now, we study certain operators over $\mathscr{A}$, where $\mathscr{A}=\{\mathfrak{A}:|\mathfrak{A}|=A\}$.

Definition 11 Let $\{\mathfrak{A}\}_{n\in\omega}$ be $a\leq$-chain of $\mathscr{A}$, i.e., $\mathfrak{A}_0\leq\mathfrak{A}_1\leq\cdots$. We say that $\mathfrak{A}$ is the supremum of $\{\mathfrak{A}_n\}_{n\in\omega}$ iff $\mathfrak{A}_n\leq\mathfrak{A}$ for all $n\in\omega$ and for any $\mathfrak{B}\in\mathscr{A}$ if $\mathfrak{A}_n\leq\mathfrak{B}$ for all n, then $\mathfrak{A}\leq\mathfrak{B}$. $\mathfrak{A}$ is denoted by sup $\{\mathfrak{A}_n\}$.

Proposition 12 If $\{\mathfrak{A}_n\}_{n\in\omega}$ is $a\leq$-chain of $\mathscr{A}$, then $\{\mathfrak{A}_n\}_{n\in\omega}$ has a supremum.

Proof Let $\mathfrak{A}_n=\langle A,\{C_i^{\mathfrak{A}_n}\},\ i\in I,\ \{R_j^{\mathfrak{A}_n}\},j\in J\rangle$ We define $\mathfrak{A}=\langle A,\{C_i^{\mathfrak{A}}\}_{i\in I},\{R_j^{\mathfrak{A}}\}_{j\in J}\rangle$ as follows: $C_i^{\mathfrak{A}}=C_i^{\mathfrak{A}_0}$ for all $i\in I$, let R_j be K-ary. For $\bar{a}\in A^k$,

$$R_j^{\mathfrak{A}}(a)=\begin{cases}1 & \text{for some } n\in\omega,\ R_j^{\mathfrak{A}_n}(\bar{a})=1\\ 0 & \text{for some } n\in\omega,\ R_j^{\mathfrak{A}_n}(\bar{a})=0\\ \uparrow & \text{for all } n\in\omega,\ R_j^{\mathfrak{A}_n}(\bar{a})\uparrow.\end{cases}$$

It is easy to verify $\mathfrak{A}=\sup\{\mathfrak{A}_n\}$.

Definition 13 Let G be an operator over $\mathscr{A}$. G is said to be momotonic iff $\mathfrak{A}\leq\mathfrak{B}\Rightarrow G(\mathfrak{A})\Rightarrow G(\mathfrak{B})$. G is said to be continuous iff for any $\leq$-chain$\{\mathfrak{A}_n\}_{n\in\omega}$, $G(\sup\{\mathfrak{A}_n\})=\sup\{G(\mathfrak{A}_n)\}$.

Theorem 14 Let G be a continuous operator over $\mathscr{A}$, then G has a least fixpoint.

Proof Let $\Lambda=\langle A,\{a_i\}_{i\in I},\{R_j^{\Lambda}\}_{j\in J}\rangle$ where $a_i\in A$ for all $i\in I$ and R_j^{Λ} is a nowhere defined function. Let $\mathfrak{A}_0=\Lambda$, $\mathfrak{A}_{n+1}=G(\mathfrak{A}_n)$. We show $\mathfrak{A}_n\leq G(\mathfrak{A}_n)$ by induction on ω. Because $\Lambda\leq G(\Lambda)\Rightarrow\mathfrak{A}_0\leq G(\mathfrak{A}_0)$ and $\mathfrak{A}_n\leq G(\mathfrak{A}_n)\Rightarrow G(\mathfrak{A}_n)\leq G^2(\mathfrak{A}_n)\Rightarrow\mathfrak{A}_{n+1}\leq G(\mathfrak{A}_{n+1})$ hence $\mathfrak{A}_n\leq G(\mathfrak{A}_n)$ for all $n\in\omega$, so $\{\mathfrak{A}_n\}_{n\in\omega}$ is $a\leq$-chain. By Pro. 12, $\{\mathfrak{A}_n\}$ has a supremum $\mathfrak{A}$. Because G is continuous, $G(\mathfrak{A})=G(\sup\{\mathfrak{A}_n\})=\sup\{G(\mathfrak{A}_n)\}=\sup\{\mathfrak{A}_{n+1}\}=\mathfrak{A}$, hence $\mathfrak{A}$ is a fixed point of G. In addition, if $G(\mathfrak{B})=\mathfrak{B}$, then we have $\mathfrak{A}_n\leq\mathfrak{B}$ for all $n\in\omega$ (easy to prove it by induction on ω), so $\sup\{\mathfrak{A}_n\}\leq\mathfrak{B}$ i.e. $\mathfrak{A}\leq\mathfrak{B}$, hence $\mathfrak{A}$ is the least fixpoint of G.

References

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部分模型之间的一些代数关系

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摘 要

我们把部分模型定义成三值逻辑的模型．对于Kleene语义，我们讨论部分模型之间的一些代数关系，并用图方法刻划了这些关系．证明了部分模型上某些算子的不动点定理．文中所讨论的代数关系对发展三值逻辑的模型起一定的作用．