Jonrnal of Mathematical Research and Exposition
Vol. 11 No.2, May, 1991

Convergence Criteria for Hermite-Fejer Interpolation*

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Abstract

In this paper we give useful convergence criteria for Hermite-Fejer interpolation $H_n(f, x)$. One of them is: $H_n(f, x)$ converges uniformly to $f(x)$ if and only if the norms of $H_n(f, x)$ are uniformly bounded for all n and $H_n(x^k, x)$ converges uniformly to x^k for $k=1,2$.

1 Introduction

Consider a system of nodes

$$-1 \leq x_{n1} < x_{n2} < \cdots < x_{nn} \leq 1, \quad n=1,2,\cdots$$

and in the following for simplicity write x_i for x_{ni}. The Hermite-Fejer interpolation $H_n(f, x)$ of degree $2n-1$ satisfying

$$H_n(f, x_i) = f(x_i), \quad H_n'(f, x_i) = 0, \quad i=1,\cdots,n$$

is given by

$$H_n(f, x) = \sum_{i=1}^{n} f(x_i) A_i(x),$$

where

$$A_i(x) = \left(1 - \frac{w''(x_i)}{w'(x_i)}(x - x_i)\right) l_i^2(x), \quad i=1,\cdots,n;$$

$$l_i(x) = \frac{w(x)}{(x - x_i) w'(x_i)}, \quad i=1,\cdots,n;$$

$$w(x) = (x - x_1)\cdots(x - x_n).$$

Denote

$$B_i(x) = (x - x_1) l_i^2(x), \quad i=1,\cdots,n.$$

In this paper we give useful convergence criteria for $H_n(f, x)$ to $f \in C[-1, 1]$. This is the following ($\Rightarrow$: = converges uniformly on $[-1,1]$ to):

Theorem For the Hermite-Fejer interpolation the following statements are equivalent:

(a) $H_n(f, x) \rightrightarrows f(x) \quad (n \to \infty), \quad \forall f \in C[-1,1]$ (1)

* Received Oct. 16, 1989.

(b) $\sum_{i=1}^{n} |A_i(x)| \leq C = \text{const}$. (2)

and

$$H_n(x^k, x) \Rightarrow x^k \quad (n \to \infty), \quad k = 1, 2; \tag{3}$$

(c) (2) holds and

$$\sum_{i=1}^{n} (x - x_i)^k A_i(x) \Rightarrow 0 \quad (n \to \infty), \quad k = 1, 2; \tag{4}$$

(d) (2) holds and

$$\sum_{i=1}^{n} x_i^{k-1} B_i(x) \Rightarrow 0 \quad (n \to \infty), \quad k = 1, 2; \tag{5}$$

(e) (2) holds and

$$\sum_{i=1}^{n} (x - x_i)^{k-1} B_i(x) \Rightarrow 0 \quad (n \to \infty), \quad k = 1, 2. \tag{6}$$

We see that this result is very similar to the famous Korovkin theorem for positive operators although $H_n(f, x)$ is not necessarily a positive operator. In Section 2 we shall give an auxiliary lemma which is of independent interest. In Section 3 we shall complete the proof of this theorem. Finally, in Section 4, we shall deduce some interesting corollaries.

2 An auxiliary lemma

Lemma

$$\sum_{i=1}^{n} (x - x_i)^k A_i(x) = k \sum_{i=1}^{n} (x - x_i)^{k-1} B_i(x), \quad 1 \leq k \leq 2n-1. \tag{7}$$

Proof We need the identities

$$\sum_{i=1}^{n} A_i(x) = 1 \tag{8}$$

and

$$\sum_{i=1}^{n} x_i^k A_i(x) + k \sum_{i=1}^{n} x_i^{k-1} B_i(x) = x^k, \quad 1 \leq k \leq 2n-1. \tag{9}$$

Hence

$$\begin{aligned}
\sum_i (x - x_i)^k A_i(x) &= \sum_i A_i(x) \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{k-j} x_i^j = x^k + \sum_{j=1}^{k} (-1)^j \binom{k}{j} x^{k-j} \sum_i x_i^j A_i(x) \\
&= x^k + \sum_{j=1}^{k} (-1)^j \binom{k}{j} x^{k-j} \left(x^j - j \sum_i x_i^{j-1} B_i(x)\right) \\
&= x^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} - \sum_{j=1}^{k} (-1)^j j \binom{k}{j} x^{k-j} \sum_i x_i^{j-1} B_i(x) \\
&= k \sum_i B_i(x) \sum_{j=1}^{k} (-1)^{j-1} \binom{k-1}{j-1} x^{k-j} x_i^{j-1} = k \sum_i (x - x_i)^{k-1} B_i(x).
\end{aligned}$$

Remark The conclusions for $k=1$ and $k=2$ are given by [1] and [2] respectively.

3 Proof

(a) $\Rightarrow$ (b) By the Banach theorem [3, p. 216].

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(b) ⇒ (c) (3), together with (8), implies (4).

(c) ⇒ (e) By the lemma.

(e) ⇔ (d) Trivial.

(e) ⇒ (a) It follows from (6) that for any fixed $k \geq 3$

$$\begin{aligned} |\sum_i x_i^{k-1} B_i(x)| &= |x^{k-1}\sum_i B_i(x) - \sum_i (x^{k-1} - x_i^{k-1}) B_i(x)| \\ &\leq |\sum_i B_i(x)| + \sum_i |x^{k-2} + x^{k-3}x_i + \cdots + x_i^{k-2}|(x - x_i) B_i(x) \\ &\leq |\sum_i B_i(x)| + (k-i)\sum_i (x - x_i) B_i(x) \Rightarrow 0 \quad (n \to \infty). \end{aligned}$$

Thus in fact (6) implies by (9) that

$$\sum_i x_i^k A_i(x) \Rightarrow x^k \quad (n \to \infty), \quad k = 1, 2, 3, \cdots$$

Using the Banach theorem mentioned above we conclude (a).

4 Corollaries

From the theorem and the Korovkin theorem for positive operators we immediately obtain.

Corollary 1 For normal systems of nodes, i.e., $A_i(x) \geq 0$, $i = 1, \cdots, n$, (1) is valid if and only if one of the following equivalent conditions is satisfied:

(a) $\sum_{i=1}^{n} (x - x_i)^2 A_i(x) \Rightarrow 0, \quad (n \to \infty)$;

(b) $\sum_{i=1}^{n} |x - x_i| A_i(x) \Rightarrow 0, \quad (n \to \infty)$;

(c) $\sum_{i=1}^{n} (x - x_i)^2 l_i^2(x) \Rightarrow 0, \quad (n \to \infty)$.

Proof Obviously we need only show the equivalence of (a) and (b). By Cauchy Inequality we see that

$$\frac{1}{2}\sum (x - x_i)^2 A_i(x) \leq \sum |x - x_i| A_i(x) \leq (\sum A_i(x) \sum (x - x_i)^2 A_i(x))^{\frac{1}{2}}$$
$$= (\sum (x - x_i)^2 A_i(x))^{\frac{1}{2}}.$$

Corollary 2 If (2) is satisfied and

$$\sum_{i=1}^{n} |x - x_i| l_i^2(x) \Rightarrow 0, \quad (n \to \infty)$$

then (1) holds.

The proof is trivial and this result generalizes Theorem 2 under the assumption of normlization in [1].

Corollary 3

$$\sum_{i=1}^{n} (x - x_i)^2 l_i^2(x) \leq 2 \sum_{i=1}^{n} |A_i(x)|$$

and especially for normal systems of nodes

$$\sum_{i=1}^{n} (x - x_i)^2 l_i^2(x) \leq 2.$$

Proof From (7') with $k=2$.

Corollary 4 For normal systems of nodes and for any $f\in C[-1,1]$,

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)l_i^2(x)=f(x),\quad -1<x<1$$

and the convergence is uniform in the interval $-1+t\leq x\leq 1-t$, where $t>0$.

Proof In [1] it is shown that for normal systems of nodes

$$\lim_{n\to\infty}\sum_{i=1}^{n} l_i^2(x)=1,\quad -1<x<1$$

and

$$\lim_{n\to\infty}\sum_{i=1}^{n}|x-x_i|l_i^2(x)=0,\quad -1<x<1$$

and, further, both the convergences are uniform in the interval $-1+t\leq x\leq 1-t$. Thus by the Korovkin theorem the conclusion of the corollary is valid.

Remark In [1] the same conclusion is obtained only for strongly normal systems of nodes.

References

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Hermite-Fejér插值的收敛准则

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摘　　要

本文给出Hermite-Fejér插值的若干收敛准则．其中之一是：Hermite-Fejer插值算子对每一个连续函数一致收敛当且仅当该算子范数一致有界且该算子对两个单项式 x 及 x^2 一致收敛．