Journal of Mathematical Research and Exposition
Vol. 11, No.2, May, 1991.

On the Degree of Approximation of Continuous Functions by Linear Means of Fourier-Jacobi Series*

Li Zhongkai

(Inst. of Applied Math., Dalian Univ. of Tech.)

§ 1 Introduction

The problem of approximation of continuous functions by Fourier-Jacobi series has been discussed by D. P. Gupta and S. M. Mazhar in [1] and Sun Xie-hua in [2]. In [6], we have obtained the degree of approximation of continuous functions by the Fejér sum of Fourier-Jacobi series. In this paper, a more general method of approximation is considered, and the above result is extended.

1.1 The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for fixed $\alpha>-1$, $\beta>-1$, may be defined on $-1\le x\le 1$ by the expansion

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, z^n = 2^{\alpha+\beta} R^{-1}(1-z+R)^{-\alpha}(1+z+R)^{-\beta}, \quad |z|<1,$$

where $R=(1-2xz+z^2)^{\frac{1}{2}}$. It is well known that $P_n^{(\alpha,\beta)}(x)$ are orthogonal on $[-1,1]$ with weight $d\sigma(x)=(1-x)^{\alpha}(1+x)^{\beta}dx$. If $f(x)\in L([-1,1];d\sigma)$, the Fourier-Jacobi expansion is

$$f(x)\sim\sum_{n=0}^{\infty}\lambda_n^{\frac{1}{2}} a_n P_n^{(\alpha,\beta)}(x), \tag{1.1}$$

where $a_n=\lambda_n^{\frac{1}{2}}\int_{-1}^{1} f(t)\,P_n^{(\alpha,\beta)}(t)\,d\sigma(t)$ are the Fourier-Jacobi coefficients of $f(x)$, and $\lambda_n^{\frac{1}{2}}$ is the constant of orthonomalization of $P_n^{(\alpha,\beta)}(x)$. The partial sum of Fourier-Jacobi series is

$$S_n^{(\alpha,\beta)}(f;x)=\sum_{k=0}^{n}\lambda_k^{\frac{1}{2}} a_k P_k^{(\alpha,\beta)}(x)=\int_{-1}^{1} f(t)\,K_n^{(\alpha,\beta)}(x,t)\,d\sigma(t),$$

where $K_n^{(\alpha,\beta)}(x,t)=\sum_{k=0}^{n}\lambda_k P_k^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(t)$ is the kernel function of the operator $S_n^{(\alpha,\beta)}(\cdot;x)$, The Fejér sum of Fourier-Jacobi series of $f(x)$ is defined by

$$F_n^{(\alpha,\beta)}(f;x)=\frac{1}{n+1}\sum_{k=0}^{n} S_k^{(\alpha,\beta)}(f;x)=\int_{-1}^{1} f(t)\,W_n^{(\alpha,\beta)}(x,t)\,d\sigma(t),$$

where

* Received July 18, 1989.

$$W_n^{(\alpha,\beta)}(x,t)=\frac{1}{n+1}\sum_{j=0}^{n}K_j^{(\alpha,\beta)}(x,t)=\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right)\lambda_k P_k^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(t)$$

is the Fejér-Jacobi kernel function.

1.2 A series $\sum_{n=0}^{\infty}c_n$ with the sequence of partial sums $\{r_n\}$ is said to be summable to s by a regular triangular matrix method $(\wedge)$, defined by Hardy [8], such that $\wedge_{n,k}\geq 0$ for all $k\leq n$ and $\wedge_{n,k}=0$, $k>n$, also $\sum_k \wedge_{n,k}\to 1$ as $n\to\infty$. write

$$t_n=\sum_{k=0}^{\infty}\wedge_{n,k}r_k\to s \text{ as } n\to\infty.$$

We call t_n the $(\wedge)$-means of $\sum_{n=0}^{\infty}c_n$.

§ 2 Results

2.1 The following result can be found in [6].

Theorem 1 Let

$$\gamma=\begin{cases}\alpha, & 0\leq x<1\\ \beta, & -1<x<0\end{cases} \tag{2.1}$$

Then for any $f(x)\in C[-1,1]$ and $n\geq 2$,

$$f(x)-F_n^{(\alpha,\beta)}(f;x)=\begin{cases}O(1)\dfrac{(1-x^2)^{-\frac{3}{2}}}{n}\sum\limits_{k=1}^{n}\omega\left(f;\dfrac{1-x^2}{k}\right), & |\gamma|\leq\dfrac{1}{2};\\ O(1)\dfrac{(1-x^2)^{-\frac{|\gamma|}{2}-\frac{5}{4}}}{n}\sum\limits_{k=1}^{n}\omega\left(f;\dfrac{1-x^2}{k}\right), & |\gamma|>\dfrac{1}{2},\end{cases} \tag{2.2}$$

where $\omega(f;t)$ is the modulus of continuity of $f(x)$.

The next theorem is proved here.

Theorem 2 Let $f(x)\in C[-1,1]$. Then for $-1<x<1$ and $n\geq 2$, the degree of approximation by the triangular matrix-means of Fourier-Jacobi series of f is given by

$$f(x)-T_n^{(\alpha,\beta)}(f;x)=f(x)(1-D_{n,n})+\begin{cases}O(1)(1-x^2)^{-\frac{3}{2}}\sum\limits_{k=1}^{n}\dfrac{D_{n,k}}{k}\omega\left(f;\dfrac{1-x^2}{k}\right), & |\gamma|\leq\dfrac{1}{2};\\ O(1)(1-x^2)^{-\frac{|\gamma|}{2}-\frac{5}{4}}\sum\limits_{k=1}^{n}\dfrac{D_{n,k}}{k}\omega\left(f;\dfrac{1-x^2}{k}\right), & |\gamma|>\dfrac{1}{2},\end{cases} \tag{2.3}$$

where $\omega(f;t)$ is the modulus of continuity of $f(x)$, γ is defined by (2.1), and $T_n^{(\alpha,\beta)}(f;x)$ are the $(\wedge)$-means such that

$$D_{n,k}=\sum_{r=0}^{k}\wedge_{n,r}.$$

Also, we define the sequence $\wedge_{n,(u)}$ in terms of $\{\wedge_{n,k}\}$, so that $\wedge_{n,(u)}$ is monotonic decreasing for all $u\geq 0$, i.e. $\wedge_{n,u}=\wedge_{n,(u)}$.

2.2 Let $\{P_n\}$ be a sequence of positive constants such that

$$P_n=p_0+p_1+\cdots+p_n\to\infty \quad \text{as} \quad n\to\infty.$$

The Nörlund means of series (1.1) are

$$N_n^{(\alpha,\beta)}(f;\,x)=\frac{1}{P_n}\sum_{k=0}^{n}p_kS_k^{(\alpha,\beta)}(f;\,x).$$

It is clear that Theorem 2 implies the following

Corollary 1 Let $p_0>0$ and $\{p_n\}$ be monotonic decreasing. Then for $f(x)\in C[-1,1]$ and $-1<x<1$,

$$f(x)-N_n^{(\alpha,\beta)}(f;\,x)=\begin{cases} O(1)\dfrac{(1-x^2)^{-\frac{3}{2}}}{P_n}\displaystyle\sum_{k=1}^{n}\frac{P_k}{k}\omega\left(f;\frac{1-x^2}{k}\right), & |\gamma|\le\frac{1}{2}; \\ O(1)\dfrac{(1-x^2)^{-\frac{|\gamma|}{2}-\frac{5}{4}}}{P_n}\displaystyle\sum_{k=1}^{n}\frac{P_k}{k}\omega\left(f;\,\frac{1-x^2}{k}\right), & |\gamma|>\frac{1}{2}, \end{cases} \tag{2.4}$$

where γ is given by (2.1).

Choose $\wedge_{a,k}=\dfrac{A_{n-k}^{\lambda-1}}{A_n^{\lambda}}$, where $A_n^{\lambda}=\dfrac{\Gamma(n+\lambda+1)}{\Gamma(n+1)\Gamma(\lambda+1)}$, For any real λ, the Cesáro means of order λ of series (1.1) are defined by

$$C_{n,\lambda}^{(\alpha,\beta)}(f;\,x)=\frac{1}{A_n^{\lambda}}\sum_{k=0}^{n}A_{n-k}^{\lambda-1}S_k^{(\alpha,\beta)}(f;\,x).$$

Since $A_{n-k}^{\lambda-1}$ is monotonic decreasing as $k=0,1,\cdots,n$ for $\lambda\ge1$, by Theorem 2 we have

Corollary 2 Let $f(x)\in C[-1,1]$. Then for $-1<x<1$ and $n\ge2$,

$$f(x)-C_{n,\lambda}^{(\alpha,\beta)}(f;\,x)=\begin{cases} O(1)\dfrac{(1-x^2)^{-\frac{3}{2}}}{A_n^{\lambda}}\displaystyle\sum_{k=1}^{n}\frac{\overline{D}_{n,k}}{k}\omega\left(f;\,\frac{1-x^2}{k}\right), & |\gamma|\le\frac{1}{2}; \\ O(1)\dfrac{(1-x^2)^{-\frac{|\gamma|}{2}-\frac{5}{4}}}{A_n^{\lambda}}\displaystyle\sum_{k=1}^{n}\frac{\overline{D}_{n,k}}{k}\omega\left(f;\frac{1-x^2}{k}\right), & |\gamma|>\frac{1}{2}, \end{cases} \tag{2.5}$$

where $\overline{D}_{n,k}=\sum_{i=0}^{k}A_{n-i}^{\lambda-1}$ and γ is defined by (2.1).

Since $\overline{D}_{n,k}\le(k+1)A_n^{\lambda-1}=\dfrac{\lambda(k+1)}{n+\lambda}A_n^{\lambda}$ for $\lambda\ge1$, by (2.5) it follows that

$$f(x)-C_{n,\lambda}^{(\alpha,\beta)}(f;\,x)=\begin{cases} O(1)\dfrac{(1-x^2)^{-\frac{3}{2}}}{n}\displaystyle\sum_{k=1}^{n}\omega\left(f;\frac{1-x^2}{k}\right), & |\gamma|\le\frac{1}{2}; \\ O(1)\dfrac{(1-x^2)^{-\frac{|\gamma|}{2}-\frac{5}{4}}}{n}\displaystyle\sum_{k=1}^{n}\omega\left(f;\frac{1-x^2}{k}\right), & |\gamma|>\frac{1}{2}. \end{cases} \tag{2.6}$$

when $\lambda=1$, we obtain (2.2) by (2.6).

In the whole paper, denote by "O" the numbers which are only dependent of α and β and independent of x,t and n, but may be different in different occurrences.

§ 3 Lemmas

We shall need the following lemmas.

Lemma 1 For the Fourier-Jacobi kernal function, we have

$$K_n^{(\alpha,\beta)}(x,t)=\begin{cases} O(n)(1-x^2)^{-\frac{\gamma}{2}-\frac{1}{4}}(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}, & \gamma\geq-\frac{1}{2}; \\ O(n)(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}, & -1<\gamma<-\frac{1}{2} \end{cases} \tag{3.1}$$

and for $-1<x, t<1$ and $t\neq x$

$$K_n^{(\alpha,\beta)}(x,t)=\begin{cases} O(1)\dfrac{(1-x^2)^{-\frac{\gamma}{2}-\frac{1}{4}}(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}}{x-t}, & \gamma\geq-\frac{1}{2}, \\ O(1)\dfrac{(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}}{x-t}, & -1<\gamma<-\frac{1}{2}, \end{cases} \tag{3.2}$$

where γ is defined by (2.1).

The proof of (3.1) and (3.2) can easily be completed by using (2.1)—(2.7) in [6].

Lemma[6] **2** For the Fejer-Jacobi kernel function, we have for $-1<x, t<1$ and $t\neq x$

$$W_n^{(\alpha,\beta)}(x,t)=\begin{cases} O(1)\dfrac{(1-x^2)^{-\frac{\gamma}{2}-\frac{1}{4}}(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}}{n(x-t)^2}, & \gamma\geq-\frac{1}{2}, \\ O(1)\dfrac{(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}}{n(x-t)^2}, & -1<\gamma<-\frac{1}{2}. \end{cases} \tag{3.3}$$

where γ is defined by (2.1).

Lemma 3 If the sequence $\{\wedge_{n,k}\}$ is defined as in Theorem 2, then

$$\omega(f;\frac{1-x^2}{n})=O(1)\sum_{k=1}^{n}\frac{D_{n,k}}{k}\omega(f;\frac{1-x^2}{k}). \tag{3.4}$$

The proof of Lemma 3 is similar to that of Lemma 1 in [7].

Lemma 4 If the sequece $\{\wedge_{n,k}\}$ is defined as in Theorem 2, then for $-1<x, t<1$

$$\sum_{j=0}^{n}\wedge_{n,j}K_j^{(\alpha,\beta)}(x,t)=\begin{cases} O(n)(1-x^2)^{-\frac{\gamma}{2}-\frac{1}{4}}(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}, & \gamma\geq-\frac{1}{2}, \\ O(n)(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}, & -1<\gamma<-\frac{1}{2}, \end{cases} \tag{3.5}$$

and for $-1<x,\ t<1$ and $|x-t|\geq\frac{1\pm x}{n}$

$$\sum_{j=0}^{n}\wedge_{n,j}K_j^{(\alpha,\beta)}(x,t)=\begin{cases}O(1)(1-x^2)^{-\frac{\gamma}{2}-\frac{1}{4}}(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}D_{n,\left(\frac{1\pm x}{|x-t|}\right)}\dfrac{(1\pm x)^{-1}}{x-t} & \gamma\geq-\dfrac{1}{2};\\ O(1)(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}D_{n,\left(\frac{1\pm x}{|x-t|}\right)}\dfrac{(1\pm x)^{-1}}{x-t} & -1<\gamma<-\dfrac{1}{2},\end{cases}\tag{3.6}$$

where γ is defined by (2.1).

Proof (3.5) can easily be obtained by (3.1). Now we prove (3.6). We only prove (3.6) for $\gamma\geq-\frac{1}{2}$. By the same method the proof for $-1<\gamma<-\frac{1}{2}$ can be finished.

Choose $m=$ integral part of $\frac{1-x}{|x-t|}$ and suppose that $|x-t|\geq\frac{1-x}{n}$. Now

$$\sum_{j=0}^{n}\wedge_{n,j}K_j^{(\alpha,\beta)}(x,t)=\sum_{j=0}^{m}\wedge_{n,j}K_j^{(\alpha,\beta)}(x,t)+\sum_{j=m+1}^{n}\wedge_{n,j}K_j^{(\alpha,\beta)}(x,t).\tag{3.7}$$

By (3.2), we have

$$\sum_{j=0}^{m}\wedge_{n,j}K_j^{(\alpha,\beta)}(x,t)=O(1)(1-x^2)^{-\frac{\gamma}{2}-\frac{1}{4}}(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}\frac{D_{n,m}}{x-t}\tag{3.8}$$

For the second part of (3.7), we sum by parts and obtain

$$\sum_{j=m+1}^{n}\wedge_{n,j}K_j^{(\alpha,\beta)}(x,t)=\wedge_{n,n}(n+1)W_n^{(\alpha,\beta)}(x,t)-\wedge_{n,m+1}(m+1)W_m^{(\alpha,\beta)}(x,t).$$
$$+\sum_{j=m+1}^{n-1}(\wedge_{n,j}-\wedge_{n,j+1})(j+1)W_j^{(\alpha,\beta)}(x,t).$$

By monotonic decreasing property of $\wedge_{n,j}$ and (3.3), we get

$$\begin{aligned}&\sum_{j=m+1}^{n}\wedge_{n,j}K_j^{(\alpha,\beta)}(x,t)\\ &=O(1)\frac{(1-x^2)^{-\frac{\gamma}{2}-\frac{1}{4}}(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}}{(x-t)^2}\left(\wedge_{n,n}+\wedge_{n,m+1}+\sum_{j=m+1}^{n-1}(\wedge_{n,j}-\wedge_{n,j+1})\right)\\ &=O(1)(1-x^2)^{-\frac{\gamma}{2}-\frac{1}{4}}(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}\frac{\wedge_{n,m}}{(x-t)^2}\\ &=O(1)(1-x^2)^{-\frac{\gamma}{2}-\frac{1}{4}}(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}\frac{m\wedge_{n,m}}{m(x-t)^2}\\ &=O(1)(1-x^2)^{-\frac{\gamma}{2}-\frac{1}{4}}(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}\frac{(1-x)^{-1}D_{n,m}}{x-t}\end{aligned}\tag{3.9}$$

From (3.7)—(3.9), it follows that

$$\sum_{j=0}^{n}\triangle_{n,j}K_j^{(\alpha,\beta)}(x,t)=O(1)(1-x^2)^{-\frac{\gamma}{2}-\frac{1}{4}}(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}\frac{(1-x)^{-1}D_{n,\left(\frac{1-x}{|x-t|}\right)}}{x-t}\tag{3.10}$$

If we choose $m=$ integral part of $\frac{1+x}{|x-t|}$ and suppose that $|x-t|\geq\frac{1+x}{n}$, by the same method we obtain

$$\sum_{j=0}^{n}\wedge_{n,j}K_j^{(\alpha,\beta)}(x,t)=O(1)(1-x^2)^{-\frac{\gamma}{2}-\frac{1}{4}}(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}\frac{(1+x)^{-1}D_{n,\left(\frac{1+x}{|x-t|}\right)}}{x-t} \tag{3.11}$$

Thus we complete the proof of Lemma 4.

§ 4 The Proof of Theorem 2

Since

$$f(x)-T_n^{(\alpha,\beta)}(f;x)=f(x)(1-D_{n,n})+\int_{-1}^{1}(f(x)-f(t))\sum_{j=0}^{n}\wedge_{n,j}K_j^{(\alpha,\beta)}(x,t)\,d\sigma(t)$$

$$=f(x)(1-D_{n,n})+O(1)\int_{-1}^{1}\omega(f;|x-t|)\left|\sum_{j=0}^{n}\wedge_{n,j}K_j^{(\alpha,\beta)}(x,t)\right|d\sigma(t),$$

it is enough if we can prove

$$\int_{-1}^{1}\omega(f;|x-t|)\left|\sum_{j=0}^{n}\wedge_{n,j}K_j^{(\alpha,\beta)}(x,t)\right|d\sigma(t)$$

$$=\begin{cases} O(1)(1-x^2)^{-\frac{3}{2}}\sum_{k=1}^{n}\frac{D_{n,k}}{k}\omega\left(f;\frac{1-x^2}{k}\right), & |\gamma|\leq\frac{1}{2};\\ O(1)(1-x^2)^{-\frac{|\gamma|}{2}-\frac{5}{4}}\sum_{k=1}^{n}\frac{D_{n,k}}{k}\omega\left(f;\frac{1-x^2}{k}\right), & |\gamma|>\frac{1}{2}. \end{cases} \tag{4.1}$$

We only prove (4.1) for $|\gamma|\leq\frac{1}{2}$. By the same method, the proof for $|\gamma|>\frac{1}{2}$ can be completed. Write

$$I\triangleq\int_{-1}^{1}\omega(f;|x-t|)\left|\sum_{j=0}^{n}\wedge_{n,j}K_j^{(\alpha,\beta)}(x,t)\right|d\sigma(t)$$

$$=\int_{-1}^{-1+\frac{1+x}{2}}+\int_{-1+\frac{1+x}{2}}^{x-\frac{1+x}{n}}+\int_{x-\frac{1+x}{n}}^{x+\frac{1-x}{n}}+\int_{x+\frac{1-x}{n}}^{1-\frac{1-x}{2}}$$

$$+\int_{1-\frac{1-x}{2}}^{1}\omega(f;|x-t|)\left|\sum_{j=0}^{n}\wedge_{n,j}K_j^{(\alpha,\beta)}(x,t)\right|d\sigma(t)$$

$$\triangleq I_1+I_2+I_3+I_4+I_5. \tag{4.2}$$

Assume that $0\leq x<1$, then $\gamma=\alpha$. When $-1<x<0$, the proof is similar. For the first part of (4.2), by (3.11) we get

$$I_1=O(1)(1-x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}\int_{-1}^{-1+\frac{1+x}{2}}\frac{\omega(f;|x-t|)}{|x-t|}D_{n,\left(\frac{1+x}{|x-t|}\right)}(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{\frac{\beta}{2}-\frac{1}{4}}dt.$$

When $-1<t\leq-1+\frac{1+x}{2}$, it is clear that

$$\frac{1+x}{2}\leq |x-t|\leq 1+x,\ \ 1\leq 1-t\leq 2.$$

Hence

$$I_1=O(1)(1-x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}\left(\frac{1+x}{2}\right)^{-1}\omega(f;1+x)D_{n,2}\int_{-1}^{-1+\frac{1+x}{2}}(1+t)^{\frac{\beta}{2}-\frac{1}{4}}dt$$

$$=O(1)(1-x^2)^{-\frac{\alpha}{2}-\frac{5}{4}}\omega(f;1-x^2)D_{n,1}, \quad (\text{as } D_{n,2}\leq 2D_{n,1}). \tag{4.3}$$

Similarly, for I_2 we have

$$I_2=O(1)(1-x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}\int_{-1+\frac{1+x}{2}}^{x-\frac{1+x}{n}}\frac{\omega(f;|x-t|)}{|x-t|}D_{n,\left(\frac{1+x}{|x-t|}\right)}(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{\frac{\beta}{2}-\frac{1}{4}}dt.$$

Since $0\leq x<1$ and $n\geq 2$, for $-1+\frac{1+x}{2}\leq t\leq x-\frac{1+x}{n}$ we have

$$\frac{1}{2}\leq 1+t<2 \ \text{ and } \ 1-x<1-t<2.$$

And so

$$I_2=O(1)(1-x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}(1-x)^{\frac{\alpha}{2}-\frac{1}{4}}\int_{-1+\frac{1+x}{2}}^{x-\frac{1+x}{n}}\frac{\omega(f;|x-t|)}{|x-t|}D_{n,\left(\frac{1+x}{|x-t|}\right)}dt$$

$$=O(1)(1-x^2)^{-\frac{1}{2}}\int_{\frac{1+x}{n}}^{\frac{1+x}{2}}\frac{\omega(f;u)}{u}D_{n,\left(\frac{1+x}{u}\right)}du.$$

Let $u=\frac{1+x}{v}$. Then

$$\int_{\frac{1+x}{n}}^{\frac{1+x}{2}}\frac{\omega(f;u)}{u}D_{n,\left(\frac{1+x}{u}\right)}du$$

$$=\int_2^n\omega\left(f;\frac{1+x}{v}\right)\frac{D_{n,(v)}}{v}dv=O(1)\sum_{k=2}^{n-1}\frac{D_{n,k+1}}{k}\omega\left(f;\frac{1+x}{k}\right)$$

$$=O(1)(1-x^2)^{-1}\sum_{k=2}^{n-1}\frac{D_{n,k}}{k}\omega\left(f;\frac{1-x^2}{k}\right),\quad (\text{as } D_{n,k+1}\leq 2D_{n,k}), \tag{4.4}$$

Thus, we have

$$I_2=O(1)(1-x^2)^{-\frac{3}{2}}\sum_{k=2}^{n-1}\frac{D_{n,k}}{k}\omega\left(f;\frac{1-x^2}{k}\right). \tag{4.5}$$

Using (3.5), we obtain an estimate for I_3, namely

$$I_3=O(n)(1-x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}\int_{x-\frac{1+x}{n}}^{x+\frac{1-x}{n}}\omega(f;|x-t|)(1-t)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{-\frac{\beta}{2}-\frac{1}{4}}d\sigma(t)$$

$$=O(n)(1-x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}\omega(f;2/n)\int_{x-\frac{1+x}{n}}^{x+\frac{1-x}{n}}(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{\frac{\beta}{2}-\frac{1}{4}}dt.$$

As $0\leq x<1$ and $n\geq 2$, for $x-\frac{1+x}{n}\leq t\leq x+\frac{1-x}{n}$ we have $\frac{1}{2}\leq 1+t\leq 2$. Consequently,

$$I_3=O(n)(1-x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}\omega(f;2/n)\int_{x-\frac{1+x}{n}}^{x+\frac{1-x}{n}}(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}dt$$

$$=O(n)(1-x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}\omega(f;2/n)\left(1-x-\frac{1-x}{n}\right)^{\frac{\alpha}{2}-\frac{1}{4}}\frac{2}{n}=O(1)(1-x^2)^{-\frac{3}{2}}\omega\left(f;\frac{1-x^2}{n}\right).$$

By Lemma 3, we get

$$I_3 = O(1)(1-x^2)^{-\frac{3}{2}}\sum_{k=1}^{n}\frac{D_{n,k}}{k}\omega(f;\frac{1-x^2}{k}). \tag{4.6}$$

To estimate I_4, we use (3.10) and obtain

$$I_4 = O(1)(1-x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}(1-x)^{-1}\int_{x+\frac{1-x}{n}}^{1-\frac{1-x}{2}}\frac{\omega(f;|x-t|)}{|x-t|}D_{n,(\frac{1-x}{|x-t|})}(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{\frac{\beta}{2}-\frac{1}{4}}dt.$$

For $x+\frac{1-x}{n}\leq t\leq 1-\frac{1-x}{2}$,

$$1<1+t<2 \text{ and } \frac{1-x}{2}\leq 1-t<1-x,$$

then

$$I_4 = O(1)(1-x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}(1-x)^{\frac{\alpha}{2}-\frac{5}{4}}\int_{x+\frac{1-x}{n}}^{1-\frac{1-x}{2}}\frac{\omega(f;|x-t|)}{|x-t|}D_{n,(\frac{1-x}{|x-t|})}dt$$

$$= O(1)(1-x^2)^{-\frac{3}{2}}\int_{\frac{1-x}{n}}^{\frac{1-x}{2}}\frac{\omega(f;u)}{u}D_{n,(\frac{1-x}{u})}du.$$

Similar to (4.4), we have

$$\int_{\frac{1-x}{n}}^{\frac{1-x}{2}}\frac{\varphi(f;u)}{u}D_{n,(\frac{1-x}{u})}du = O(1)\sum_{k=2}^{n-1}\frac{D_{n,k}}{k}\omega(f;\frac{1-x^2}{k}).$$

Consequently,

$$I_4 = O(1)(1-x^2)^{-\frac{3}{2}}\sum_{k=2}^{n-1}\frac{D_{n,k}}{k}\omega(f;\frac{1-x^2}{k}). \tag{4.7}$$

For the last part of (4.2), we get by (3.10)

$$I_5 = O(1)(1-x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}(1-x)^{-1}\int_{1-\frac{1-x}{2}}^{1}\frac{\omega(f;|x-t|)}{|x-t|}D_{n,(\frac{1-x}{|x-t|})}(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}(1+t)^{\frac{\beta}{2}-\frac{1}{4}}dt$$

When $1-\frac{1-x}{2}\leq t<1$, we have

$$\frac{1-x}{2}\leq|x-t|<1-x \text{ and } 1\leq 1+t<2.$$

Hence

$$I_5 = O(1)(1-x^2)^{-\frac{\alpha}{2}-\frac{1}{4}}(1-x)^{-2}\omega(f;1-x)D_{n,2}\int_{1-\frac{1-x}{2}}^{1}(1-t)^{\frac{\alpha}{2}-\frac{1}{4}}dt$$

$$= O(1)(1-x^2)^{-\frac{3}{2}}\omega(f;1-x^2)D_{n,1}. \tag{4.8}$$

From (4.2), (4.3) and (4.5)—(4.8), it follows that for $|y|\leq\frac{1}{2}$,

$$I = O(1)(1-x^2)^{-\frac{3}{2}}\sum_{k=1}^{n}\frac{D_{n,k}}{k}\omega(f;\frac{1-x^2}{k}).$$

The proof of Theorem 2 is completed.

References

[1] D. P. Gupta and S. M. Mazhar, Approx. Th. and Appl., 2: 3 (1986), 1—6.

[2] Sun Xiehua, Approx. Th. and Appl., 4: 3 (1988), 79—85.

[3] G. Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publication, Vol. XXIII (1967).

[4] T. Singh, J. Indian Math. Soc. (N. S), 45 (1981), No. 1—4, 353—359.

[5] P. L. Butzer and R. J. Nessel, Fourier Analysis and Approximation, Vol. 1, Chinese Translation, translated by Zheng Weixing et al.

[6] Li Zhongkai, Approx. Th. and Appl., 6: 3 (1990), 30—41.

[7] P. D. Kathal, A. S. B. Holland and B. N. Sahney, Acta Math. Acad. Sci. Hung., 30 (3—4) (1977), 227—231.

[8] G. H. Hardy, Divergent Series (Oxford, 1949).

Fourier-Jacobi级数的线性平均对连续函数的逼近度

李 中 凯

(大连理工大学应用数学研究所)

摘 要

本文研究了Fourier-Jacobi级数的一般线性求和问题，得到了其对连续函数的点态逼近阶，所得结果是文献[6]中关于Fourier-Jacobi 级数的Fejér和的结果的直接延伸.同时得到了Fourier-Jacobi级数的λ阶Cesaro平均($\lambda\geq 1$)和Nörlund平均对连续函数的逼近度.