Journal of Mathematical Research and Exposition
Vol.11, No.2, May, 1991

# On a Class of Univalent Functions*

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**Abstract**

A sharp coefficient estimate, distortion theorem and the radius of convexity are determined for the class $R(\alpha,\beta,A,B)$ of function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ satisfying the condition

$$\left|\frac{f'(z)-1}{Bf'(z)-[B+(A-B)(1-\alpha)]}\right|<\beta$$

for some $\alpha,\beta\,(0\le\alpha<1,\ 0<\beta\le1)$ and $-1\le A<B\le1,\ 0<B\le1$ and for all $z$ in $U=\{z:|z|<1\}$. A sufficient condition for a function to belong to $R(\alpha,\beta,A,B)$ has also been determined.

## I. Introduction

Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be analytic in a convex domain $D$. If $f(z)$ satisfies the condition

$$\mathrm{Re}(f'(z))>0 \tag{1.1}$$

for all $z$ in $D$, then it is well known (see [6], [9] etc.) that $f(z)$ is univalent in $D$.

Let $T$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.2}$$

which are analytic in the unit disc $U=\{z:|z|<1\}$ MacGregor [4] investigated the properties of functions $f(z)\in T$, satisfying $\mathrm{Re}(f'(z))>0$ for $z\in U$. In [1] Caplinger and Causey, and in [7] Padmanabhan extended the results of MacGregor [4] by introducing the class $R(\beta)$ of functions $f(z)\in T$ and satisfying for all $z$ in $U$ the condition

$$\left|\frac{f'(z)-1}{f'(z)+1}\right|<\beta \tag{1.3}$$

for some $\beta$, $0<\beta\le1$.

* Received Aug. 14, 1989.

Also in [3] Juneja and Mogra studied the class $R(\alpha,\beta)$ of functions $f(z)\in T$ satisfying for all $z$ in $U$ the condition

$$\left|\frac{f'(z)-1}{f'(z)+1-2\alpha}\right|<\beta \tag{1.4}$$

for some $\alpha,\beta(0\leq\alpha<1,\ 0<\beta\leq1)$.

In this paper, we consider the class $R(\alpha,\beta,A,B)$ of functions $f(z)\in T$, satisfying for all $z$ in $U$ the condition

$$\left|\frac{f'(z)-1}{Bf'(z)-[B+(A-B)(1-\alpha)]}\right|<\beta \tag{1.5}$$

for some $\alpha,\beta(0\leq\alpha<1,\ 0<\beta\leq1)$ and $-1\leq A<B\leq1,\ 0<B\leq1$.

We note that $R(\alpha,\beta,-1,1)=R(\alpha,\beta)$, and the class $R(\alpha,\beta,A,B)$ is a subclass of the class of functions whose derivatives have a positive real part in $U$ and hence a function in $R(\alpha,\beta,A,B)$ is univalent in $U$. It is easily seen that for $f(z)\in R(\alpha,\beta,A,B)$, the values $f'(z)$ lie inside the circle in the right half plane with center $\dfrac{1-[B+(A-B)(1-\alpha)]B\beta^2}{1-B^2\beta^2}$ and radius $\dfrac{(B-A)\beta(1-\alpha)}{1-B^2\beta^2}$. Further it follows from Schwarz's Lemma [5] that if $f(z)\in R(\alpha,\beta,A,B)$ then

$$f'(z)=\frac{1+[B+(A-B)(1-\alpha)]\beta z\varphi(z)}{1+B\beta z\varphi(z)},$$

where $\varphi(z)$ is analytic and $|\varphi(z)|\leq1$ in $U$.

We obtain a sharp coefficient estimate, distortion theorem, radius of convexity, etc. for $f(z)\in R(\alpha,\beta,A,B)$. Our results yield for $A=-1$ and $B=1$ the corresponding results obtained by Juneja and Mogra [3], yield for $A=-1$, $B=1$ and $\alpha=0$ the corresponding results obtained by Caplinger and Causey [1] and Padmanabhan [7]. For $A=-1$, $B=1$ and $\beta=1$, these give the results for the class of functions satisfying $\mathrm{Re}(f'(z))>\alpha$ for all $z\in U$ which generalize the corresponding results of MacGregor [4]. We also obtain a sufficient condition for a function to be in $R(\alpha,\beta,A,B)$.

## 2. A coefficient formula

**Theorem 1** If $f(z)\in T$ is in $R(\alpha,\beta,A,B)$ for some $\alpha,\beta(0\leq\alpha<1,\ 0<\beta\leq1)$ and $-1\leq A<B\leq1,\ 0<B\leq1$, then

$$|a_n|\leq\frac{(B-A)(1-\alpha)\beta}{n},\quad n\geq2.$$

The bound is sharp.

**Proof** Since $f(z)\in R(\alpha,\beta,A,B)$, we have

$$f'(z)=\frac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)}, \tag{2.1}$$

where $w(z)=\sum_{k=1}^{\infty}t_kz^k=z\varphi(z)$ is analytic and satisfies the condition $|w(z)|<\beta$ for

$z\in U$. Then (2.1) gives

$$(Bf'(z)-[B+(A-B)(1-\alpha)])w(z)=1-f'(z)$$

or

$$[(B-A)(1-\alpha)+B\sum_{k=2}^{\infty}ka_kz^{k-1}][\sum_{k=1}^{\infty}t_kz^k]=-\sum_{k=2}^{\infty}ka_kz^{k-1}. \quad (2.2)$$

Equating corresponding coefficients on both sides of (2.2) we observe that the coefficient $a_n$ on the right of (2.2) depends only on $a_2, a_3, \cdots, a_{n-1}$ on the left of (2.2). Hence for $n\geq 2$, it follows from (2.2) that

$$[(B-A)(1-\alpha)+B\sum_{k=2}^{n-1}ka_kz^{k-1}]w(z)=-\sum_{k=2}^{n}ka_kz^{k-1}-\sum_{k=n+1}^{\infty}b_kz^{k-1}.$$

Since $|w(z)|<\beta$, we get

$$\beta|(B-A)(1-\alpha)+B\sum_{k=2}^{n-1}ka_kz^{k-1}|\geq|\sum_{k=2}^{n}ka_kz^{k-1}+\sum_{k=n+1}^{\infty}b_kz^{k-1}|. \quad (2.3)$$

Squaring both sides of (2.3) and integrating round $|z|=r$, $0<r<1$, we obtain

$$\beta^2[(B-A)^2(1-\alpha)^2+B^2\sum_{k=2}^{n-1}k^2|a_k|^2r^{2k-2}]\geq\sum_{k=2}^{n}k^2|a_k|^2r^{2k-2}+\sum_{k=n+1}^{\infty}|b_k|^2r^{2k-2}.$$

If we take the limit as $r$ approaches 1, then

$$\beta^2[(B-A)^2(1-\alpha)^2+B^2\sum_{k=2}^{n-1}k^2|a_k|^2]\geq\sum_{k=2}^{n}k^2|a_k|^2$$

or

$$(1-B^2\beta^2)\sum_{k=2}^{n-1}k^2|a_k|^2+n^2|a_n|^2\leq(B-A)^2\beta^2(1-\alpha)^2.$$

Since $0<\beta\leq 1$ and $0<B\leq 1$,

$$n^2|a_n|^2\leq(B-A)^2\beta^2(1-\alpha)^2,$$

it follows that

$$|a_n|\leq\frac{(B-A)(1-\alpha)\beta}{n}, \quad n\geq 2.$$

The bounds are sharp for the functions

$$f(z)=\int_0^z\frac{1-[B+(A-B)(1-\alpha)]\beta z^{n-1}}{1-B\beta z^{n-1}}dz$$

for $n\geq 2$ and $z\in U$.

### 3. A sufficient condition for a function to be in $R(\alpha, \beta, A, B)$

**Theorem 2** Let $f(z)\in T$. If for some $\alpha, \beta$ $(0\leq\alpha<1,\ 0<\beta\leq 1)$ and $-1\leq A<B\leq 1$, $0<B\leq 1$,

$$\sum_{n=2}^{\infty}(1+B\beta)n|a_n|\leq(B-A)(1-\alpha)\beta, \quad (3.1)$$

then $f(z)$ belongs to $R(\alpha, \beta, A, B)$.

**Proof** We emply the same technique as used by Clunie and Keogh [2].

Thus suppose that (3.1) holds and that $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, then for $|z|<1$,

$$|f'(z)-1|-\beta|Bf'(z)-[B+(A-B)(1-\alpha)]|$$
$$=|\sum_{n=2}^{\infty}na_nz^{n-1}|-\beta|(B-A)(1-\alpha)+B\sum_{n=2}^{\infty}na_nz^{n-1}|\leq\sum_{n=2}^{\infty}n|a_n|r^{n-1}-\beta\{(B-A)(1-\alpha)$$
$$-B\sum_{n=2}^{\infty}n|a_n|r^{n-1}\}<\sum_{n=2}^{\infty}n|a_n|-(B-A)(1-\alpha)\beta+B\beta\sum_{n=2}^{\infty}n|a_n|$$
$$=\sum_{n=2}^{\infty}(1+B\beta)n|a_n|-(B-A)(1-\alpha)\beta\leq 0.$$

Hence it follows that for $z\in U$:

$$\left|\frac{f'(z)-1}{Bf'(z)-[B+(A-B)(1-\alpha)]}\right|<\beta\ ,$$

therefore $f(z)\in R(\alpha,\beta,A,B)$.

We note that

$$f(z)=z-\frac{(B-A)(1-\alpha)\beta}{(1+B\beta)n}z^n$$

is an extremal function from the above theorem since

$$\left|\frac{f'(z)-1}{Bf'(z)-[B+(A-B)(1-\alpha)]}\right|=\beta$$

for $z=1$, $0\leq\alpha<1$, $0<\beta\leq 1$, $-1\leq A<B\leq 1$, $0<B\leq 1$, and $n=2,3,\cdots$. We also observe that the converse to the theorem is false in that

$$f(z)=\frac{[B+(A-B)(1-\alpha)]}{B}z-\frac{(B-A)(1-\alpha)}{B^2\beta}\log(1-B\beta z)\in R(\alpha,\beta,A,B)$$

but

$$\sum_{n=2}^{\infty}\frac{(1+B\beta)}{(B-A)(1-\alpha)\beta}|a_n|=\sum_{n=2}^{\infty}\frac{(1+B\beta)n}{(B-A)(1-\alpha)\beta}\cdot\frac{(B-A)(1-\alpha)}{n}B^{n-2}\beta^{n-1}$$
$$=\sum_{n=2}^{\infty}(1+B\beta)(B\beta)^{n-2}>1$$

for $\alpha,\beta,A,B$ satisfying $0\leq\alpha<1$, $0<\beta\leq 1$ and $-1\leq A<B\leq 1$, $0<B\leq 1$.

## 4. Distortion Theorem

**Theorem 3** Let the function $f(z)$ defined by (1.2) be in the class $R(\alpha,\beta,A,B)$. Then

$$|f(z)|\leq\frac{[B+(A-B)(1-\alpha)]}{B}|z|-\frac{(B-A)(1-\alpha)}{B^2\beta}\log(1-B\beta|z|), \tag{4.1}$$

$$|f(z)|\geq\frac{[B+(A-B)(1-\alpha)]}{B}|z|+\frac{(B-A)(1-\alpha)}{B^2\beta}\log(1+B\beta|z|). \tag{4.2}$$

All the estimates are sharp.

**Proof** Since $f(z)\in R(\alpha,\beta,A,B)$, we observe that the condition (1.5) coupled with an application of Schwarz's Lemma [5], implies

$$|f'(z)-\xi|<R,$$

where

$$\xi=\frac{1-[B+(A-B)(1-\alpha)]B\beta^2r^2}{1-B^2\beta^2r^2},$$

and

$$R=\frac{(B-A)(1-\alpha)\beta r}{1-B^2\beta^2r^2},\quad (|z|=r).$$

Hence we have

$$\frac{1+[B+(A-B)(1-\alpha)]\beta r}{1+B\beta r}\leq \mathrm{Re}(f'(z))\leq\frac{1-[B+(A-B)(1-\alpha)]\beta r}{1-B\beta r} \tag{4.3}$$

Let

$$g(z)=\frac{1-[B+(A-B)(1-\alpha)]\beta z}{1-B\beta z}.$$

Since $g(0)=1=f'(0)$ and $g(z)$ is univalent in $U$, it follows that $f'$ is subordinate to $g$. Hence

$$|f'(z)|\leq\frac{1-[B+(A-B)(1-\alpha)]\beta r}{1-B\beta r}. \tag{4.4}$$

In view of

$$|f(z)|=\left|\int_0^s f(s)\,ds\right|\leq\int_0^{|z|}|f'(te^{i\theta})|\,dt,$$

and with the aid of (4.4) we may write

$$|f(z)|\leq\int_0^{|z|}\frac{1-[B+(A-B)(1-\alpha)]\beta t}{1-B\beta t}\,dt$$

$$=\frac{[B+(A-B)(1-\alpha)]}{B}|z|-\frac{(B-A)(1-\alpha)}{B^2\beta}\log(1-B\beta|z|).$$

Further, by using (4.3) we may write

$$|f(z)|\geq\int_0^{|z|}\mathrm{Re}[f'(te^{i\theta})]\,dt\geq\int_0^{|z|}\frac{1+[B+(A-B)(1-\alpha)]\beta t}{1+B\beta t}\,dt$$

$$=\frac{[B+(A-B)(1-\alpha)]}{B}|z|+\frac{(B-A)(1-\alpha)}{B^2\beta}\log(1+B\beta|z|).$$

For the function $f(z)=\dfrac{[B+(A-B)(1-\alpha)]}{B}z-\dfrac{(B-A)(1-\alpha)}{B^2\beta}\log(1-B\beta z)$ which satisfies the condition $\left|\dfrac{f'(z)-1}{Bf'(z)-[B+(A-B)(1-\alpha)]}\right|<\beta$ for $z\in U$. the equality is attained in (4.1).

For the function $f(z)=\dfrac{[B+(A-B)(1-\alpha)}{B}z+\dfrac{(B-A)(1-\alpha)}{B^2\beta}\log(1+B\beta z)$ which also belongs to $R(\alpha,\beta,A,B)$, the equality is attained in (4.2).

## 5. The radius of convexity for functions in the class $R(\alpha、\beta, A, B)$

Let $B$ denote the class of analytic functions $w(z)$ in $|z|<1$ which satisfy the conditions (i) $w(0)=0$ and (ii) $|w(z)|<1$ for $|z|<1$. For obtaining the radius of convexity for functions in the class $R(\alpha,\beta,A,B)$, we require the following lemmas.

**Lemma 1**[8] If $w(z)\in B$, then for $|z|<1$,

$$|zw'(z)-w(z)|\leq\frac{|z|^2-|w(z)|^2}{1-|z|^2}. \tag{5.1}$$

**Lemma 2** For $w(z)\in B$, we have

$$\mathrm{Re}\left\{\frac{zw'(z)}{(1+B\beta w(z))(1+[B+(A-B)(1-\alpha)]\beta w(z))}\right\}$$

$$\leq\frac{-1}{(B-A)^2(1-\alpha)^2\beta^2}\mathrm{Re}\left\{B\beta p(z)+\frac{[B+(A-B)(1-\alpha)]\beta}{p(z)}-[2B+(A-B)(1-\alpha)]\beta\right\}$$
$$+\frac{r^2|B\beta p(z)-[B+(A-B)(1-\alpha)]\beta|^2-|1-p(z)|^2}{(B-A)^2(1-\alpha)^2\beta^2(1-r^2)|p(z)|}, \tag{5.2}$$

where $p(z)=\dfrac{1+[B+(A-B)(1-\alpha)]\beta w(z)}{1+B\beta w(z)}$, $r=|z|$ and $0\leq\alpha<1$, $0<\beta\leq1$, $-1\leq A<B\leq1$, $0<B\leq1$.

**Proof** Since $p(z)=\dfrac{1+[B+(A-B)(1-\alpha)]\beta w(z)}{1+B\beta w(z)}$, we have

$$w(z)=\frac{1-p(z)}{(B\beta p(z)-[B+(A-B)(1-\alpha)]\beta)}. \tag{5.3}$$

After simple calcululations, we get

$$\frac{1}{(1+B\beta w(z))(1+[B+(A-B)(1-\alpha)]\beta w(z)]}$$
$$=\frac{(B\beta p(z)-[B+(A-B)(1-\alpha)]\beta)^2}{(B-A)^2(1-\alpha)^2\beta^2 p(z)} \tag{5.4}$$

and

$$\frac{w(z)}{(1+B\beta w(z))(1+[B+(A-B)(1-\alpha)]\beta w(z))}$$
$$=\frac{-1}{(B-A)^2(1-\alpha)^2\beta^2}\left\{B\beta p(z)+\frac{[B+(A-B)(1-\alpha)]\beta}{p(z)}\right.$$
$$\left.-[2B+(A-B)(1-\alpha)]\beta\right\}. \tag{5.5}$$

Therefore using Lemma 1, equations (5.4) and (5.5) give

$$\left|\frac{zw'(z)}{(1+B\beta w(z))(1+[B+(A-B)(1-\alpha)]\beta w(z))}\right.$$
$$\left.-\frac{w(z)}{(1+B\beta w(z))(1+[B+(A-B)(1-\alpha)]\beta w(z))}\right|$$
$$\leq\frac{r^2|B\beta p(z)-[B+(A-B)(1-\alpha)]\beta|^2-|1-p(z)|^2}{(B-A)^2(1-\alpha)^2\beta^2(1-r^2)|p(z)|}$$

or

$$\mathrm{Re}\left\{\frac{zw'(z)}{(1+B\beta w(z))(1+[B+(A-B)(1-\alpha)]\beta w(z))}\right\}$$
$$\leq\mathrm{Re}\left\{\frac{w(z)}{(1+B\beta w(z))(1+[B+(A-B)(1-\alpha)]\beta w(z))}\right\}$$
$$+\frac{r^2|B\beta p(z)-[B+(A-B)(1-\alpha)]\beta|^2-|1-p(z)|^2}{(B-A)^2(1-\alpha)^2\beta^2(1-r^2)|p(z)|}. \tag{5.6}$$

Hence the lemma follows immediately from (5.5) and (5.6).

**Remark** The transformation $p(z)=\dfrac{1+[B+(A-B)(1-a)]\beta w(z)}{1+B\beta w(z)}$ maps the circle $|w(z)|\leq r$ onto the circle

$$\left|p(z)-\frac{1+[B+(A-B)(1-a)]B\beta^2 r^2}{1-B^2\beta^2 r^2}\right|\leq\frac{(B-A)(1-a)\beta r}{1-B^2\beta^2 r^2}.$$

**Theorem 4** Let $f(z)\in R(a,\beta,A,B)$, $0\leq a<1$, $0<\beta\leq 1$ and $-1\leq A<B\leq 1$, $0<B\leq 1$. Then $f$ is convex in $|z|<r_0$, where $r_0$ is the smallest positive root of

$$\sqrt{(1+B\beta)(1-B\beta r^2)(1+[B+(A-B)(1-a)]\beta)(1-[B+(A-B)(1-a)]\beta r^2)}$$
$$-(1-[B+(A-B)(1-a)]B\beta^2 r^2)-[B+(A-B)(1-a)]\beta(1-r^2)=0. \quad (5.7)$$

if $R_0\geq R_1$ and $r_0$ is the smallest positive root of

$$1+2[B+(A-B)(1-a)]\beta r+[B+(A-B)(1-a)]B\beta^2 r^2=0 \quad (5.8)$$

if $R_0\leq R_1$, where

$$R_0=\left\{\frac{1+[B+(A-B)(1-a)]\beta)(1-[B+(A-B)(1-a)]\beta r^2}{(1+B\beta)(1-B\beta r^2)}\right\}^{\frac{1}{2}} \quad (5.9)$$

and

$$R_1=\frac{1+[B+(A-B)(1-a)]\beta r}{1+B\beta r}, \quad |z|=r<1. \quad (5.10)$$

All the above estimates are sharp.

**Proof** Since $f(z)\in R(a,\beta,A,B)$, we have

$$f'(z)=\frac{1+[B+(A-B)(1-a)]\beta w(z)}{1+B\beta w(z)} \quad (5.11)$$

where $w(z)\in B$. Differentiating (5.11) logarithmically we have

$$1+\frac{zf''(z)}{f'(z)}=1-\frac{(B-A)(1-a)\beta zw'(z)}{(1+B\beta w(z))(1+[B+(A-B)(1-a)]\beta w(z))}.$$

An application of Lemma 2 to the above equation gives

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\geq\frac{1}{(B-A)(1-a)\beta}\Big[\mathrm{Re}\left\{B\beta p(z)+\frac{[B+(A-B)(1-a)]\beta}{p(z)}\right\}$$
$$-\frac{r^2|B\beta p(z)-[B+(A-B)(1-a)]\beta|^2-|1-p(z)|^2}{(1-r^2)|p(z)|}\Big]-\frac{2[B+(A-B)(1-a)]}{(B-A)(1-a)}. \quad (5.12)$$

By setting $p(z)=a+m+in$, $R^2=(a+m)^2+n^2$, where

$$a=\frac{1-[B+(A-B)(1-a)]B\beta^2 r^2}{1-B^2\beta^2 r^2}$$

and denoting the expression on the right hand side of (5.12) by $E(m,n)$, we get

$$E(m,n)=\frac{-2[B+(A-B)(1-a)]}{(B-A)(1-a)}+\frac{1}{(B-A)(1-a)\beta}\Big[B\beta(a+m)$$
$$+[B+(A-B)(1-a)]\beta(a+m)R^{-2}-\frac{(1-B^2\beta^2 r^2)}{(1-r^2)}(d^2-m^2-n^2)R^{-1}\Big], \quad (5.13)$$

where $d=\dfrac{(B-A)(1-a)\beta r}{1-B^2\beta^2 r^2}$

Differentiating (5.13) partially with respect to $n$, we get

$$\frac{\partial E(m,n)}{\partial n}=\frac{1}{(B-A)(1-a)\beta}nR^{-4}F(m,n) \tag{5.14}$$

where

$$F(m,n)=-2\beta[B+(A-B)(1-a)](a+m)$$
$$+\frac{(d^2-m^2-n^2)(1-B^2\beta^2r^2)}{(1-r^2)}R+2\frac{1-B^2\beta^2r^2}{1-r^2}R^3 .$$

It is easy to see that $F(m,n)>0$ and so (5.14) gives that the minimum of $E(m,n)$ inside the circle $m^2+n^2\le d^2$ is attained on the diameter $n=0$. Hence putting $n=0$ in (5.13), we get

$$M(R)=E(m,0)=\frac{1}{(B-A)(1-a)\beta}\Big[\Big(B\beta+\frac{1-B^2\beta^2r^2}{1-r^2}\Big)R$$
$$+\frac{(1+[B+(A-B)(1-a)]\beta)(1-[B+(A-B)(1-a)]\beta r^2)}{(1-r^2)}R^{-1}$$
$$-2a\frac{1-B^2\beta^2r^2}{1-r^2}\Big]-\frac{2[B+(A-B)(1-a)]}{(B-A)(1-a)}$$

where $R=a+m$ and $a-d\le R\le a+d$. Thus the absolute minimum of $M(R)$ in $(0,\infty)$ is attained at

$$R_0=\Big\{\frac{(1+[B+(A-B)(1-a)]\beta)(1-[B+(A-B)(1-a)]\beta r^2)}{(1+B\beta)(1-B\beta r^2)}\Big\}^{\frac{1}{2}} \tag{5.15}$$

and equals

$$M(R_0)=\frac{2}{(B-A)(1-a)\beta(1-r^2)}$$
$$\times\{\sqrt{(1+B\beta)(1-B\beta r^2)(1+[B+(A-B)(1-a)]\beta)(1-[B+(A-B)(1-a)]\beta r^2)}$$
$$-(1-[B+(A-B)(1-a)]B\beta^2r^2)-[B+(A-B)(1-a)]\beta(1-r^2)\}. \tag{5.16}$$

It is easy to see that $R_0<a+d$, but $R_0$ is not always greater than $a-d$, In such a case when $R_0\notin[a-d,a+d]$, the minimum of $M(R)$ on the segment $[a-d,a+d]$ is attained at $R_1=a-d$ and equals

$$M(R_1)=M(a-d)$$
$$=\frac{1+2[B+(A-B)(1-a)]\beta r+[B+(A-B)(1-a)]B\beta^2r^2}{(1+B\beta r)(1+[B+(A-B)(1-a)]\beta r)} . \tag{5.17}$$

It follows from what has been said that the bound $r_0$ of convexity for the class $R(a,\beta,A,B)$ is determined from the equation $M(R_0)=0$ or from the equation $M(R_1)=0$. Also, $M(R_0)=M(R_1)$ for such values of $a\,(0\le a<1)$, $\beta\,(0<\beta\le1)$ and $-1\le A<B\le1$, $0<B\le1$ for which $R_0=R_1$.

From (5.12), (5.16) and (5.17), we have

$$\mathrm{Re}\Big\{1+\frac{zf''(z)}{f'(z)}\Big\}$$

$$
\geq \begin{cases} \dfrac{2}{(B-A)(1-a)\beta(1-r^2)} \\ \times \left\{ \sqrt{(1+B\beta)(1-B\beta r^2)(1+[B+(A-B)(1-a)]\beta)(1-[B+(A-B)(1-a)]\beta r^2)} \right. \\ \left. -(1-[B+(A-B)(1-a)]B\beta^2 r^2)-[B+(A-B)(1-a)]\beta(1-r^2) \right\} \\ \text{if } R_0 \geq R_1, \\ \dfrac{1+2[B+(A-B)(1-a)]\beta r+[B+(A-B)(1-a)]B\beta^2 r^2}{(1+B\beta r)(1+[B+(A-B)(1-a)]\beta r)} \\ \text{if } R_0 \leq R_1. \end{cases} \tag{5.18}
$$

Now the theorem follows easily from (5.18). The functions given by

$$
f'(z)=\frac{1+[B+(A-B)(1-a)]\beta z}{1+B\beta z} \tag{5.19}
$$

and

$$
f'(z)=\frac{1-\{1+[B+(A-B)(1-a)]\beta\}bz+[B+(A-B)(1-a)]\beta z^2}{1-(1+B\beta)bz+B\beta z^2}, \tag{5.20}
$$

where $b$ is determined by the relation

$$
\frac{1-\{1+[B+(A-B)(1-a)]\beta\}br+[B+(A-B)(1-a)]\beta r^2}{1-(1+B\beta)br+B\beta r^2}=R_0, \tag{5.21}
$$

show that the results obtained in the Theorem are sharp.

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