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Some Classes of Differential Subordinations and Their Applications*

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Abstract

Let G be univalent in the unit disc Δ. Let $\psi: C^2\times\Delta\to C$ be analytic in a domain $D\subset C$ and let $p(z)$ be analytic in Δ with $(p(z), zp'(z); z)\in D$ when $z\in\Delta$. Assume that $p(z)$ satisfies the differential subordination

$$\psi(p(z),\ zp'(z);\ z)\prec G(z),\quad z\in\Delta,$$

(where "$\prec$" denotes subordination). We determine some conditions on ψ and G so that $\mathrm{Re}\,p(z)>\rho$ in Δ and give some applications of these resullts.

Let $\Delta=\{|z|<1\}$, $H\in(\Delta)$ be the class of analytic functions $f(z)$ in Δ and let $S(\Delta)$ be the subclass of $H(\Delta)$ consisting of functions which are univalent in Δ.

At first we prove a lemma.

Lemma Let $\Omega\subset C$, $a\in C$ with $\mathrm{Re}\,a>\tau$. Suppose that the function $\psi: C^2\times\Delta\to C$ is analytic in a domain $D\subset C$ and satisfies the condition

$$\psi(\tau+ix,\ y;\ z)\notin\Omega,\quad z\in\Delta \tag{1}$$

for all real x and $y\leq-\dfrac{1}{2\mathrm{Re}(a-\tau)}|(a-\tau)-ix|^2$.

If $p(z)\in H(\Delta)$ with $p(0)=a$ and

$$\psi(p(z),\ zp'(z);\ z)\in\Omega,\quad z\in\Delta, \tag{2}$$

then $\mathrm{Re}\,p(z)>\tau$ in Δ.

Proof Clearly, we need only to prove that

$$p(z)\prec\frac{a+(\bar a-2\tau)z}{1-z},\quad z\in\Delta. \tag{3}$$

To do this, let

$$p(z)=\frac{a+(\bar a-2\tau)\omega(z)}{1-\omega(z)},\quad z\in\Delta. \tag{4}$$

Where $\omega(z)\in H(\Delta)$ with $\omega(0)=0$ and $\omega(z)\neq1$ in Δ. We wish to show $|\omega(z)|<1$ for all $z\in\Delta$. If this is not the case, then there is a point z_0 in Δ satisfying $|z_0|=r<1$ such that

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$$\max_{|z|=r}|\omega(z)|=|\omega(z_0)|=1.$$

By Jack's Lemma, we obtain $z_0\omega'(z_0)=l\ \ \omega(z_0)$, where $l\geq 1$. Let $\omega(z_0)=e^{i\theta}(0<\theta<2\pi)$. A simple calculus yields

$$p(z_0)=\tau+ix,$$

where

$$x=\mathrm{Im}a+\frac{\sin\theta}{1-\cos\theta}\mathrm{Re}(a-\tau)\in R,$$

and

$$z_0p'(z_0)=-\frac{l}{1-\cos\theta}\mathrm{Re}(a-\tau).$$

From (2) we obtain

$$\psi(p(z_0),z_0p'(z_0);z_0)=\psi(\tau+ix,\ -\frac{l}{1-\cos\theta}\mathrm{Re}(a-\tau);z_0)\in\Omega. \qquad (5)$$

On the other hand, we have

$$-\frac{l}{1-\cos\theta}\mathrm{Re}(a-\tau)\leq-\frac{1}{1-\cos\theta}\mathrm{Re}(a-\tau)=-\frac{1}{2\mathrm{Re}(a-\tau)}|(a-\tau)-ix|^2.$$

This contradicts (1) and hence we may conclude that $|\omega(z)|<1$ in Δ. By (4), subordination (3) holds. The proof of the Lemma is complete.

For the case of $\tau=0$, another proof of this Lemma may be found in [1].

Theorem 1 Let $a\geq 0$, $p(z)\in H(\Delta)$ with $p(0)=a$ and $0<\frac{a}{2}\leq\tau<a$. If $p(z)$ satisfies the differential subordination

$$p(z)+a\frac{zp'(z)}{p(z)}\prec G(z),\quad z\in\Delta,$$

then $\mathrm{Re}\,p(z)>\tau$ in Δ. Where

$$G(z)=\frac{a+(a-2\mu)z}{1-z},\quad \mu=\tau-\frac{a(a-\tau)}{2\tau}.$$

Proof The function $G(z)$ is a conformal map of Δ onto the half-plane $\mathrm{Re}\,w>\tau-\frac{a(a-\tau)}{2\tau}$ in the W-plane.

Setting $\Omega=G(\Delta)$ and

$$\psi(p(z),\ zp'(z);z)=p(z)+a\frac{zp'(z)}{p(z)}$$

in Lemma, hence we obtain

$$\psi(p(z),zp'(z);z)\in\Omega,\quad z\in\Delta.$$

We shall show that ψ satisfies the condition (1). In fact, for all $x\in R$, $z\in\Delta$ and

$$y\leq-\frac{1}{2(a-\tau)}\{(a-\tau)^2+x^2\},$$

We have

$$\mathrm{Re}\,\psi(\tau+ix,\ y;\ z)=\tau+\frac{a\tau y}{\tau^2+x^2}\leq\tau-\frac{a(a-\tau)}{2\tau},$$

and hence the condition (1) holds, i.e. $\psi(\tau+ix,y;z)\notin\Omega$. The conclusion of Theorem 1 follows from the Lemma.

Corollary 1 Let $\gamma \geq \beta(1-2\rho)$, $\beta>0$, $0\leq\rho<1$. Let $h(z)\in H(\Delta)$ with $h(0)=1$ and satisfying

$$\operatorname{Re} h(z)>\rho-\frac{1-\rho}{2(\gamma+\beta\rho)}, \quad z\in\Delta.$$

Suppose $q(z)=1+\cdots$ be a regular solution of the differential equation

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=h(z) \tag{6}$$

in Δ, then $\operatorname{Re} q(z)>\rho$ in Δ.

Proof Setting $a=\beta+\gamma$, $\tau=\beta\rho+\gamma$, $\alpha=1$ and $p(z)=\beta q(z)+\gamma$ in Theorem 1, we have $0<\frac{a}{2}\leq\tau<a$, and $\beta h(z)+\gamma\prec\frac{a+(a-2\mu)z}{1-z}$, $(z\in\Delta)$, where $\mu=\tau-\frac{a-\tau}{2\tau}$. Together with (6), all the conditions of Theorem 1 are satisfied, and so we have $\operatorname{Re}(\beta q(z)+\gamma)>\tau$, therefore conclusion of corollary 1 holds.

By the help of the proof of Theorem 1 we can moreover prove the following theorem.

Theorem 2 Let $p(z)\in H(\Delta)$, $p(0)=a$, $\operatorname{Re} a>\tau$. If $\alpha\in R$, $\alpha\tau\geq0$ and $p(z)$ satisfies

$$P(z)+\alpha\frac{zp'(z)}{p(z)}\prec\frac{a+(\overline{a}-2\tau)z}{1-z}, \quad z\in\Delta,$$

then $\operatorname{Re} p(z)>\tau$ in Δ.

In order to prove Theorem 3, we consider the function (compare [4]):

$$Q_a(z)=2C\frac{A(z)}{1-A(z)^2}, \quad z\in\Delta, \tag{7}$$

where $\operatorname{Re} a>0$, $A(z)=(z+b)/(1+\overline{b}z)$ with

$$C=\frac{1}{\operatorname{Re} a}\left(|a|\sqrt{\alpha^2+2\alpha\operatorname{Re} a}+\alpha\operatorname{Im} a\right), \quad \alpha>0$$

and $b\in\Delta$ is defined by $a=2\,Cb/(1-b^2)$.

Since $A(z)$ is an automorphism of the disc $\overline{\Delta}$, therefore $Q_a(z)$ is univalent in Δ (i.e. $Q_a(z)\in S(\Delta)$). Clearly, $Q_a(0)=a$ and $Q_a(\Delta)$ is the complex plane with a slit along the half-lines: $\operatorname{Re} w=0$, $\operatorname{Im} w\geq C$ and $\operatorname{Re} w=0$, $\operatorname{Im} w\leq -C$.

Theorem 3 Let $\alpha>0$, $\operatorname{Re} a>0$ and $\operatorname{Im} a\leq0$. If $p(z)\in H(\Delta)$ with $p(0)=a$ and

$$p(z)+\alpha\frac{zp'(z)}{p(z)}\prec Q_a(z), \quad z\in\Delta, \tag{8}$$

then $\operatorname{Re} p(z)>0$ in Δ.

Proof If we let $\Omega=Q_a(\Delta)$ and

$$\psi(p(z),zp'(z);z)=p(z)+\alpha zp'(z)/p(z), \quad z\in\Delta$$

in Lemma, then

$$\psi(p(z),zp'(z);z)\in\Omega, \quad z\in\Delta.$$

We can prove that ψ satisfies condition (1), i.e.

$$s(x,y)=\psi(ix,y;z)=ix+\alpha y/ix\notin\Omega$$

for all $x\in R$, $z\in\Delta$ and

$$y \leq -\frac{1}{2\mathrm{Re}\,a}|a-ix|^2 = -\frac{1}{2\mathrm{Re}\,a}(x^2+|a|^2-2x\mathrm{Im}\,a).$$

In fact, if $x=0$, then $s(x, y)=\infty \notin \Omega$. If $x>0$, then $\mathrm{Re}\,s(x, y)=0$ and

$$\mathrm{Im}s(x, y)=x-\frac{ay}{x} \geq x+\frac{ax^2+a|a|^2-2ax\mathrm{Im}\,a}{2x\mathrm{Re}\,a} \geq C.$$

Similarly, for $x<0$ we deduce $\mathrm{Re}\,s(x, y)=0$ and $\mathrm{Im}s(x, y) \leq -C$. Hence conclusion of Theorem 3 follows from the Lemma.

Remark The restriction for complex number c in [4, Theorem 1] is not sufficient, because for $x<0$, one cannot deduce $\mathrm{Im}\,p(z) \geq C$ (see [4]). A similar case appeared in [4, Theorem 2] too.

By using Theorem 3 we obtain the following results.

Corollary 2 Let $p(z) \in H(\Delta)$ with $p(0)=1$. For $a>0$ and

$$p(z)+a\frac{zp'(z)}{p(z)} \prec Q_1(z)=\frac{1+z}{1-z}+\frac{2az}{1-z^2}, \quad z \in \Delta,$$

we have $p(z) \prec (1+z)/(1-z)$ in Δ.

For $a=1$, this result was proved in [2].

From Corollary 2 we easily obtain (because $C=\sqrt{a(a+2)}$).

Corollary 3 Let $a>0$ and $p(z) \in H(\Delta)$ with $p(0)=1$. If either

(i) $$\left|p(z)+a\frac{zp'(z)}{p(z)}-1\right|<1+a,$$

or

(ii) $$|\mathrm{Im}\,(p(z)+azp'(z)/p(z))|<\sqrt{a(a+2)}$$

holds for $z \in \dot{\Delta}$, then $\mathrm{Re}\,p(z)>0$ in Δ.

We now consider some applications of these Theorems and Corollaries.

Let

$$A=\{f(z):\ f \in H(\Delta),\ f(0)=1-f'(0)=0\}$$

and let Δ stands for $\Delta \setminus \{0\}$.

If $a \in R$, $\rho<1$, $\mathrm{Re}\mu>\rho$, $g(z) \in S(\Delta)$ and $f(z) \in A$ with $f(z) \cdot f'(z) \neq 0$ in Δ such that

$$a\left(1+\frac{zf''(z)}{f'(z)}\right)+(\mu-a)\frac{zf'(z)}{f(z)} \prec g(z), \quad z \in \Delta, \tag{9}$$

then we say that $f(z)$ belongs to the class $M(a, \mu, \rho; g(z))$.

For the choice of $\mu=1$ and $g(z)=\frac{1+(1-2\rho)z}{1-z}$, $M(a, \mu, \rho; g(z))$ reduces to the class of a-convex functions of order ρ. Again when $a=0$, $\mu=e^{i\lambda}$, $\lambda \in (-\pi/2, \pi/2)$, and $g(z)=\frac{e^{i\lambda}+(e^{-i\lambda}-2\rho\cos\lambda)z}{1-z}$, this yields the class of λ-spiralike functions of order ρ, and so on.

St. Ruscheweyh and V. Singh proved the following result in [3].

Theorem A Let $\mu=\beta+\gamma$, $\beta>0$, $\mathrm{Re}\gamma \geq 0$. Assume $f(z) \in M(1, \mu, \mathrm{Re}\gamma; g(z))$, where

$$g(z) \prec \frac{\mu + (\bar{\mu} - 2\mathrm{Re}\gamma)z}{1-z}, \quad z \in \Delta.$$

Then $\mathrm{Re}(\mu z f'(z)/f(z)) > \mathrm{Re}\gamma$ in Δ.

From Theorem A, one can deduce a sufficient condition related to spiralike functions, this result discussed by Eenigenburg et al in [6].

We now give a generalized version of the Theorem A.

Theorem 4 Let $a\rho \geq 0$ and

$$g(z) \prec \frac{\mu + (\bar{\mu} - 2\rho)z}{1-z}, \quad z \in \Delta.$$

Assume $f(z) \in M(a, \mu, \rho; g(z))$, then

$$\mathrm{Re}\left(\mu \frac{zf'(z)}{f(z)}\right) > \rho$$

in Δ.

Proof If we let

$$p(z) = \mu z f'(z)/f(z), \tag{10}$$

then

$$p(z) + a\frac{zp'(z)}{p(z)} \prec g(z), \tag{11}$$

the conclusion immediately follows from Theorem 2.

By Theorem 4 we obtain

Corollary 4 Let $\lambda \in (-\pi/2, \pi/2)$, $\rho < 1$, $a \in R$, $a\rho\cos\lambda \geq 0$. If

$$g(z) \prec \frac{e^{i\lambda} + (e^{-i\lambda} - 2\rho\cos\lambda)z}{1-z}, \quad z \in \Delta,$$

then

$$M(a, e^{i\lambda}, \rho\cos\lambda; g(z)) \subset S_\lambda(\rho).$$

Where $S_\lambda(\rho)$ is the class of λ-spiralike functions of order ρ.

By means of Theorem 2, 3 and combining (10) with (11) we can obtain

Theorem 5 Let $a \geq 0$, $0 \leq \rho < 1$, $\mu > \rho$. If

$$g(z) \prec \frac{\mu + (\mu - 2\rho)z}{1-z} (Q_u(z)), \quad z \in \Delta,$$

then

$$M(a, \mu, \rho; g(z)) \subset S^*\left(\frac{\rho}{\mu}\right) \; (M(a, \mu, 0; g(z)) \subset S^*(0)).$$

Where $S^*(\rho)$ is the class of starlike functions of order ρ.

Finally we discuss a class of integral operators.

Let $\beta, \gamma \in C$, $\beta \neq 0$, $f(z) \in A$, we denote

$$F(z) = J(f)(z) = \left\{\frac{\beta+\gamma}{z^\gamma}\int_0^z t^{\gamma-1} f(t)^\beta \mathrm{d}t\right\}^{1/\beta}, \quad z \in \Delta. \tag{12}$$

If we let $p(z) = \beta\frac{zF'(z)}{F(z)} + \gamma$, then by (12) we obtain

$$p(z) + \frac{zp'(z)}{p(z)} = \beta\frac{zf'(z)}{f(z)} + \gamma. \tag{13}$$

S. S. Miller et al proved the following result in [5].

Theorem B Let $\beta, \gamma \in C$, $\beta = |\beta| e^{i\lambda}$, $\lambda \in (-\pi/2, \pi/2)$. If $\rho \in R$ and satisfies $-\mathrm{Re}\gamma / \mathrm{Re}\beta \leq \rho < 1$, then the integral operator (12) satisfies

$$J: S_\lambda(\rho) \to S_\lambda(\rho).$$

This Theorem improved the corresponding result discussed by S. Bajpai in [7].

Clearly, Theorem B is an immediate corollary of the Theorem 2.

By using Theorem 3 we obtain

Theorem 6 Let $\beta = |\beta| e^{i\lambda}$, $\lambda \in (-\pi/2, \pi/2)$ and

$$\mathrm{Re}(\beta+\gamma) > 0 \geq |\beta| \rho \cos\lambda + \mathrm{Re}\gamma, \quad \rho < 1.$$

If $\mathrm{Im}(\beta+\gamma) \leq 0$, $f(z) \in A$ and

$$\beta \frac{zf'(z)}{f(z)} + \gamma \prec Q_{\beta+\gamma}(z), \quad z \in \Delta,$$

then the function $F(z)$ given by operator (12) belongs to $S_\lambda(\rho)$.

Proof From Theorem 3 and (13), we obtain $\mathrm{Re}\, p(z) > 0 \geq |\beta| \rho \cos\lambda + \mathrm{Re}\gamma$. Hence

$$\mathrm{Re}(e^{i\lambda} zF'(z)/F(z)) > \rho \cos\lambda, \quad z \in \Delta.$$

The proof is complete.

If we let $\gamma = 0$, $\beta = 1/a > 0$, then (12) becomes

$$F(z) = J(f)(z) = \{\frac{1}{a}\int_0^z t^{-1} f(t)^{1/a} dt\}^a. \tag{14}$$

Setting $p(z) = zF'(z)/F(z)$, by (14) we have

$$p(z) + a\frac{zp'(z}{p(z)} = \frac{zf'(z)}{f(z)}$$

From Theorem 2 and corollary 1, we obtain following two results, respectively.

Corollary 5 If $a > 0$, $0 \leq \rho < 1$, then the operator (14) maps $S^*(\rho)$ onto the class of a-convex functions of order ρ, the subclass of $S^*(\rho)$.

Corollary 6 If $\rho \in [1/2, 1)$, $a > 0$, then the operator (14) satisfies

$$J: S^*(\rho - \frac{a(1-\rho)}{2\rho}) \to S^*(\rho).$$

By Corollary 6 (letting $a = 1$), we obtain

Corollary 7 If $\rho \in [1/2, 1)$, $g(z) \in A$ and

$$\mathrm{Re}(1 + \frac{zg''(z)}{g'(z)}) > \rho - \frac{1-\rho}{2\rho}$$

in Δ, then $g(z) \in S^*(\rho)$.

In particular, letting $\rho = 1/2$ in corollary 7, this result is well-known. Also, if $g(z) \in A$, and $|g''(z)/g'(z)| \leq 1$ in Δ, then $g(z) \in S^*(1/2)$.

From Theorem 6 or Corollary 2 we can deduce

Corollary 8 If $a > 0$, $f(z) \in A$ and

$$\frac{zf'(z)}{f(z)} \prec Q_1(z) = \frac{1+z}{1-z} + \frac{2az}{1-z^2}, z \in \Delta,$$

then the function $F(z)$ given by (14) belongs to $S^*(0)$.

This result was recently proved in [4].

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一类微分从属性及其应用

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摘 要

记 $\Delta=\{|z|<1\}$．设函数 $\psi: \mathbf{C}^2\times\Delta\to\mathbf{C}$ 在区域 $D\subset\mathbf{C}$ 中解析，G 是 Δ 中的单叶解析函数．若 Δ 中的解析函数 $p(z)$ 满足微分从属关系

$$\psi(p(z),\ zp'(z); z)\prec G(z),\quad z\in\Delta,$$

则可确定 ψ 和 G 的某些条件使之 $\mathrm{Re}\,p(z)>\rho$ $(z\in\Delta)$，并且给出这些结果的某些应用．