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An Extension of an Inversion Theorem of Carlitz*

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For $q\neq 1$, Gauss q-coefficients may be defined as follows

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}, \quad (q)_n = \prod_{j=1}^{n} (1-q^j).$$

where $0\leq k\leq n$ and $(q)_0=1$. For $k>n$ one may define $\begin{bmatrix} n \\ k \end{bmatrix}=0$.

Theorem Let $\{a_j(z)\}$ and $\{b_j(z)\}$ be any two sequences of complex valued functions of z and let q ($q\neq 1$) be an arbitrary complex number such that

$$\varphi(x,n,q)=\prod_{j=1}^{n}(a_j(q)+q^{-x}b_j(q))\neq 0 \qquad (0)$$

for all non-negative integers x and n with $\varphi(x,0,q)=1$. Then we have the following pair of reciprocal relations

$$f(n)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}\varphi(k,n,q)\,g(k), \qquad (1)$$

$$g(n)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}\frac{a_{k+1}(q)+q^{-k}b_{k+1}(q)}{\varphi(n,k+1,q)}f(k). \qquad (2)$$

Three important special cases have to be mentioned. (i) The inversion theorem of L. Carlitz just corresponds to the case where $a_j(q)\equiv\alpha_j$ and $b_j(q)\equiv\beta_j$ are complex constants. (ii) Define $a_j(q)=\alpha_j-\beta_j/(q^{-1}-1)$ and $b_j(q)=\beta_j/(q^{-1}-1)$. Then

$$a_j(q)+q^{-x}b_j(q)=\alpha_j+\beta_j\cdot\left(\frac{q^{-x}-1}{q^{-1}-1}\right)$$

so that (1) and (2) yield a reciprocal pair just equivalent to that given by W. C. Chu (cf. Mathematica Applicata, 2(1989), No.1, 47—52). (iii) In the case (ii) letting $q\to 1$ so that

$$(a_j(q)+q^{-x}b_j(q))\to(\alpha_j+\beta_j x), \quad \begin{bmatrix} n \\ k \end{bmatrix}\to\binom{n}{k},$$

we see that (1) and (2) lead to the basic inverse relations due to H. W. Gould and L. C. Hsu (cf. Duke Math. J. 40(1973), No.4, 885—891; 893—901).

It suffices to show (1)⇒(2). This can be accomplished by using some properties of q-coefficients and by a tricky splitting of the factor

$$a_{k+1}+q^{-k}b_{k+1}=\frac{1-q^{n-k}}{1-q^{n-j}}(a_{k+1}+q^{-j}b_{k+1})+\frac{\omega(k-1,n)}{\omega(k,n)}\frac{1-q^{k-j}}{1-q^{n-j}}(a_{k+1}+q^{-n}b_{k+1}),$$

where $\omega(k,n)=q^{\frac{1}{2}(n-k)(n-k-1)}$, $a_{k+1}\equiv a_{k+1}(q)$ and $b_{k+1}\equiv b_{k+1}(q)$.

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