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Definite Conditions and Asymptotic Solution of One Nonlinear Partial Differential Equation in Morphogenesis*

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Abstract

One reaction-diffusion equation

$$\varphi_{tx}+\varepsilon\varphi_t\varphi_{xx}-\varphi_{xxx}=\gamma(\varphi_x)$$

has been presented in the study of Morphogenesis. In this paper, reasonable definite conditions of the equation are proposed and the asymptotic form of its solution is obtained by using perturbation method. So the existence of solution of this problem is solved.

Ⅰ. Physical background and its mathematical model

In a 1-dimensional thin tube containing nutrition liquid, cells will multiply and diffuse. The diffusion equation can be described by the following dimensionless differential equations

$$c_t+vc_x-c_{xx}=\gamma(c) \tag{1.1}$$

$$v_x=\varepsilon c_t\ ,\quad 0<\varepsilon\ll 1 \tag{1.2}$$

where C = mass coefficient of the cells, V = convection velocity and $\gamma(C)$ = increasing velocity. As the equations are dimensionless, we assume the length of the tube is 1. In paper [1], no definite conditions have been given and certainly any form of the solution can't be got. It is only pointed out that the most suitable boundary conditions may be free boundary.

To simplify the mathematical model, we introduce a function $\varphi(x,t)$, such that

$$c=\varphi_x\ ,\quad v=\varepsilon\varphi_t$$

and equations (1.1) and (1.2) become

$$\varphi_{tx}+\varepsilon\varphi_t\varphi_{xx}-\varphi_{xxx}=\gamma(\varphi_x) \tag{1.3}$$

Here, we make such assumptions: the interval of x is $0\le x\le b(\varepsilon t)$ $(0<b(\varepsilon t)<1,\ \lim\limits_{t\to\infty}b(\varepsilon t)=1)$ and b is a known slowly varying function of t. Else, we let the

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boundary condition at $x=0$ be $\frac{\partial c}{\partial x}\Big|_{x=0}=\frac{\partial \varphi_x}{\partial x}\Big|_{x=0}=0$, which shows the cells at $x=0$ will not diffuse inward or outward and c is knowable at free boundary $x=b(\varepsilon t)$, e.g., we let $c|_{x=b(\varepsilon t)}=\varphi_x|_{x=b(\varepsilon t)}=\varepsilon g(t)$ and also let the distribution of c at initial moment be known $\varphi_x(x,0)=f(x)$ $(0\leq x\leq b(0))$. Especially suggested is that the convection velocity v of the cell at free boundary $x=b(\varepsilon t)$ is vary small, i.e., $\varphi_t(b(\varepsilon t),t)=\varepsilon a(t)$. Thus, we have the following problem

$$
(\mathrm{I})\quad\begin{cases}
\varphi_{tx}+\varepsilon\varphi_t\varphi_{xx}-\varphi_{xxx}=\gamma(\varphi_x), \quad (0<x<b(\varepsilon t),\ t>0) & (1.4)\\
\varphi_{xx}|_{x=0}=0, \quad \varphi_x|_{x=b(\varepsilon t)}=\varepsilon g(t) & (1.5)\\
\varphi_x(x,0)=f(x), \quad (0\leq x\leq b(0)) & (1.6)\\
\varphi_t|_{x=b(\varepsilon t)}=\varepsilon a(t) & (1.7)
\end{cases}
$$

where $f(x)$, $g(t)$, $b(x)$ and $a(t)$ are all smooth.

2. Solving the definite problem (Ⅰ)

2 a. Assmuming $\gamma=a\varphi_x$, so Eq. (1.4) becomes

$$\varphi_{tx}+\varepsilon\varphi_t\varphi_{xx}-\varphi_{xxx}=a\varphi_x \quad (a<0). \tag{2.1}$$

In actual problem, the order of ε is 10^{-4}, so we use perturbation methods to solve the problem [2]. Let the solution be denoted by

$$\varphi(x,t,\varepsilon)=\varphi_0(x,t)+\varepsilon\varphi_1(x,t)+\varepsilon^2\varphi_2(x,t)+\cdots \tag{2.2}$$

Substituting (2.2) into (2.1), (1.4),(1.5),(1.6) and (1.7), also letting $b(0)=b_0$ and comparing the coefficients of equal powers of ε on both sides of equations, we obtain the following sequence of fixed boundary problems which can be solved in order

$$
(\mathrm{II})\quad\begin{cases}
\varphi_{0tx}-\varphi_{0xxx}=a\varphi_{0x} \quad (0<x<b_0,\ t>0) & (2.3)\\
\frac{\partial\varphi_{0x}}{\partial x}\Big|_{x=0}=0, \quad \varphi_{0x}|_{x=b_0}=0 & (2.4)\\
\varphi_{0x}(x,0)=f(x), \quad (0\leq x\leq b_0) & (2.5)\\
\varphi_{0t}|_{x=b_0}=0 & (2.6)
\end{cases}
$$

$$
(\mathrm{III})\quad\begin{cases}
\varphi_{1tx}-\varphi_{1xxx}=a\varphi_{1x}+h_1(x,t) \quad (0<x<b_0,\ t>0) & (2.7)\\
\varphi_{1xx}(0,t)=0,\ \varphi_{1x}(b_0,t)=p_1(t) & (2.8)\\
\varphi_{1x}(x,0)=0 \quad (0\leq x\leq b_0) & (2.9)\\
\varphi_{1t}(b_0,t)=q_1(t) & (2.10)
\end{cases}
$$

Since the definite problems of φ_2, φ_3, $\cdots$, φ_i, $\cdots$ in formal solution (2.2) have the same form as problem (Ⅲ), we will not write them out. If φ_0, φ_1, $\cdots$ in fixed boundary problem (Ⅱ), (Ⅲ), $\cdots$ are found, and they are bounded to t, (2.2) will be asymptotic solution (i.e. approximate analytic solution) of free boundary problem (Ⅰ) [2].

Now we solve problem (Ⅱ). From (2.3)~(2.5) we get

$$\varphi_{0x}=\sum_{k=0}^{\infty}A_k e^{-\left[\frac{(k+1/2)^2\pi^2}{b_0^2}-a\right]t}\cos\frac{(k+1/2)\pi x}{b_0} \tag{2.11}$$

where

$$A_k=\frac{2}{b_0}\int_0^{b_0}f(x)\cos\frac{(k+1/2)\pi x}{b_0}\,dx$$

Integrating (2.11) for x and combining (2.11) with (1.6), we have

$$\varphi_0(x,t)=\sum_{k=0}^{\infty}A_k\frac{b_0}{(k+1/2)\pi}\left[\sin\frac{(k+1/2)\pi x}{b_0}-(-1)^k\right]e^{\left[a-\frac{(k+1/2)^2\pi^2}{b_0^2}\right]t}+c_0 \tag{2.12}$$

where c_0 is an arbitrary constant.

Next we solve problem (Ⅲ). In problem (Ⅲ)

$$h_1(x,t)=-\varphi_{0t}\varphi_{0xx},\quad p_1(t)=g(t)-\varphi_{0xx}(b_0,t)b'(0)t,$$

$$q_1(t)=a(t)-\varphi_{0tx}(b_0,t)b'(0)t$$

From (2.7)~(2.9) we get

$$\varphi_{1x}=\sum_{k=0}^{\infty}\int_0^t B_k(\tau)e^{at-\frac{(k+1/2)^2\pi^2(t-\tau)}{b_0^2}}d\tau\cos\frac{(k+1/2)\pi x}{b_0}$$

$$+\sum_{k=0}^{\infty}c_k e^{\left[a-\frac{(k+1/2)^2\pi^2}{b_0^2}\right]t}\cos\frac{(k+1/2)\pi x}{b_0}+e^{at}p_1(t) \tag{2.13}$$

where $H_1(x,t)=e^{-at}h_1(x,t)$, $p_1(t)=e^{-at}p_1(t)$.

$$B_k(\tau)=\frac{2}{b_0}\int_0^{b_0}[H_1(\xi,\tau)-p_1'(\tau)]\cos\frac{(k+1/2)\pi\xi}{b_0}\,d\xi$$

$$C_k=\frac{2}{b_0}\int_0^{b_0}-p_1(0)\cos\frac{(k+1/2)\pi\xi}{b_0}\,d\xi$$

with $\varphi_{1x}=\psi e^{at}$, we integrate (2.13) with respect to x and combining with (2.10) we have

$$\varphi_1(x,t)=\int_{b_0}^{x}\psi(x,t)e^{at}dx+\int_0^t q_1(t)\,dt+c_1 \tag{2.14}$$

From (2.12) and (2.14) the first order asymptotic solution of (Ⅰ) is resulted

$$\varphi(x,t)=\varphi_0(x,t)+\varepsilon\varphi_1(x,t)+0(\varepsilon^2)$$

From above-mentioned solving process, the asymptotic solution to original problem (Ⅰ) is unique except the difference of a constant. Only if $a(t)$ and $g(t)$ satisfy some conditions, φ are bounded for t. So asymptotic solution is uniformly valid [2]. As for higher-order solution, the solving method is completely similar to that of the first order, which will not be disscussed here.

2 b. Assume $\gamma(\varphi_x)$ is a nonlinear function, differentiable in any order.

Here again substituting (2.2) into (Ⅰ), we gel the definite problem of degenerate solution $\varphi_0(x,t)$

$$(\text{Ⅳ})\quad\begin{cases}\varphi_{0tx}-\varphi_{0xxx}=\gamma(\varphi_{0x}) & (0<x<b_0,\ t>0) \quad (2.15)\\ \varphi_{0xx}(0,t)=\varphi_{0x}(b_0,t)=0 & (2.16)\\ \varphi_{0x}(x,0)=f(x), & (0\le x\le b_0) \quad (2.17)\\ \varphi_{0t}(b_0,t)=0 & (2.18)\end{cases}$$

Still let $\varphi_{0x}=\psi_0(x,t)$, we have

$$\text{(V)}\quad\begin{cases}\psi_{0t}-\psi_{0xx}=\gamma(\psi_0) & (0<x<b_0,\quad t>0) \qquad (2.19)\\ \psi_{0x}(0,t)=\psi_0(b_0,t)=0 & \qquad (2.20)\\ \psi_0(x,0)=f(x), & (0\le x\le b_0) \qquad (2.21)\end{cases}$$

As the eigenvalue and eigenfunction of the definite problem of the homogeneous equation of problem (V) are

$$\lambda_k=(k+1/2)\pi/b_0$$

$$\psi_{0k}(x)=\cos(k+1/2)\pi x/b_0 \qquad k=0,1,2,\cdots$$

and the normalized function system of (V) is

$$\overline{\psi}_{0k}(x)=\sqrt{2/b_0}\,\psi_{0k}(x)$$

So we have

$$\int_0^{b_0}\overline{\psi}_{0m}(x)\overline{\psi}_{0n}(x)\,dx=\delta_{mn} \tag{2.22}$$

Let (V) have solution $\psi_0(x,t)=\sum_{k=0}^{\infty}a_k(t)\overline{\psi}_{0k}(x)$. Substituting $\psi_0(x,t)$ into (2.19) and (2.21), we have

$$\begin{cases}\sum_{k=0}^{\infty}\left(\dfrac{da_k}{dt}+\lambda_k^2a_k\right)\overline{\psi}_{0k}(x)=\gamma\left[\sum_{k=0}^{\infty}a_k(t)\overline{\psi}_{0k}(x)\right]\\ \sum_{k=0}^{\infty}a_k(0)\overline{\psi}_{0k}(x)=f(x)\end{cases}$$

Combining with (2.22) yields

$$\begin{cases}\dfrac{da_n}{dt}+\lambda_n^2a_n=\int_0^{b_0}\gamma\left[\sum_{k=0}^{\infty}a_k(t)\overline{\psi}_{0k}(x)\right]\overline{\psi}_{0n}(x)\,dx\\ a_n(0)=\int_0^{b_0}f(x)\overline{\psi}_{0n}(x)\,dx, \qquad n=0,1,2,\cdots\end{cases}$$

Now we use successive approximation to get the solution $a_n(t)$. First, let's solve the following problem

$$\begin{cases}a_n'(t)+\lambda_n^2a_n=\int_0^{b_0}\gamma\left[\sum_{k=0}^{m}a_k(t)\overline{\psi}_{0k}(x)\right]\overline{\psi}_{0n}(x)\,dx & (2.23)\\ a_n(0)=\int_0^{b_0}f(x)\overline{\psi}_{0n}(x)\,dx, \qquad n=0,1,\cdots,m & (2.24)\end{cases}$$

Let $\quad g_n(a_1,a_2,\cdots,a_m)=\int_0^{b_0}\gamma\left[\sum_{k=0}^{m}a_k(t)\psi_{0k}(x)\right]\psi_{0n}(x)\,dx-\lambda_n^2a_n$.

Introduce vectors

$$A=\begin{bmatrix}a_0(t)\\ a_1(t)\\ \vdots\\ a_m(t)\end{bmatrix},\qquad G=\begin{bmatrix}g_0\\ g_1\\ \vdots\\ g_m\end{bmatrix},\qquad A^0=\begin{bmatrix}a_0(0)\\ a_1(0)\\ \\ a_m(0)\end{bmatrix}$$

The vector representations of (2.23) and (2.24) are

$$\begin{cases}dA/dt=G\\ A|_{t=0}=A^0\end{cases}$$

The process of successive approximations is below. Let

$$
\begin{aligned}
&A_0(t)=A^0\\
&A_1(t)=A^0+\int_0^t G(A^0)\,\mathrm{d}t=A^0+G(A^0)t\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&A_i(t)=A^0+\int_0^t G[A_{i-1}(s)]\,\mathrm{d}s \qquad (2.25)\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

It can be proved that $A_i(t)$ convergess uniformly to $A(t)$ at the interval $[0, T]$, when functions γ and f satisfy some conditions [3]. Thus, the approximate solution ot problem (V) reads

$$\psi_0=\varphi_{0x}\approx\sum_{k=0}^{m} a_k(t)\overline{\psi_{0k}}(x) \tag{2.26}$$

Combining (2.23) with (2.25) $\varphi_{0x}(b_0, t)=0$, we have

$$\varphi_0(x,t)=\int_{b_0}^{x}\psi_0(x,t)\,\mathrm{d}x+c_0 \tag{2.27}$$

where c_0 is an arbitrary constant.

Below we'll get the approximate solution of first-order asymptotic solution $\varphi_1(x,t)$. Its definite problem is

$$
(\text{VI})\quad \begin{cases}
\varphi_{1tx}-\varphi_{1xxx}=h_1(x,t)+\varphi_{1x}\gamma'(\varphi_{0x}) & (0<x<b_0,\ t>0)\\
\varphi_{1xx}(0,t)=0,\quad \varphi_{1x}(b_0,t)=p_1(t) & \\
\varphi_{1x}(x,0)=0 & (0\le x\le b_0)\\
\varphi_{1t}(b_0,t)=q_1(t) &
\end{cases} \tag{2.28}
$$

where $h_1(x,t)=-\varphi_{0t}\varphi_{0xx}$, $\quad p_1(t)=g(t)-b'(0)t\varphi_{0xx}(b_0,t)$,

$$q_1(t)=a(t)-b'(0)t\varphi_{0tx}(b_0,x).$$

Let $\varphi_{1x}=\psi_1+p_1(t)$, we bave from (VI)

$$
(\text{VII})\quad \begin{cases}
\psi_{1t}-\psi_{1xx}=H_1(\psi_1, x, t)\\
\psi_{1x}(0,t)=\psi_1(b_0,t)=0\\
\psi_1(x,0)=-p_1(0)
\end{cases}
$$

where $H_1(\psi_1, x, t)=\gamma'(\varphi_{0x})[\psi_1+p_1(t)]+h_1(x,t)$.

For problem (VII) using above variation of parameter and successive approximaiton, we obtain the approximate solutions $\overline{\psi_1}(x,t)$ and $\overline{\varphi_1}(x,t)$ of $\psi_1(x,t)$ and $\varphi_1(x,t)$ respectively, i.e.

$$\overline{\psi_1}(x,t)=\sum_{k=0}^{m}a_{1k}(t)\cos\frac{(k+1/2)\pi}{b_0}x$$

$$\overline{\varphi_1}(x,t)=\int_{b_0}^{x}[\overline{\psi_1}(x,t)+p_1(t)]\,\mathrm{d}x+\int_0^t q_1(t)\,\mathrm{d}t+c_1$$

Therefore, the asymptotic solution to problem (I) is got

$$\varphi(x,t)=\varphi_0(x,t)+\varepsilon\varphi_1(x,t)+0(\varepsilon^2)$$

Next consider a relatively special case. When the order of magnitude of $\varphi(x,t)$ is ε and $\gamma(0)=0$, it's allowed to let

$$\varphi(x,t)=\varepsilon\varphi_1(x,t)+\varepsilon^2\varphi_2(x,t)+\cdots \tag{2.29}$$

Substituting (2.29) into (I) and comparing the coefficients of like powers of ε give

ε^1

$$\begin{cases} \varphi_{1tx}-\varphi_{1xxx}=\gamma'(0)\varphi_{1x} & (0<x<b_0,\ t>0) \\ \varphi_{1xx}(0,t)=0,\ \varphi_{1x}(b_0,t)=g(t) & \\ \varphi_{1x}(x,0)=0 & (0\le x\le b_0) \\ \varphi_{1t}(b_0,t)=a(t), & \end{cases}$$

ε^2

$$\begin{cases} \varphi_{2tx}-\varphi_{2xxx}=\dfrac{\gamma''(0)}{2}\varphi_{1x}^2+\gamma'(0)\varphi_{2x} & (0<x<b_0,\ t>0) \\ \varphi_{2xx}(0,t)=0,\ \varphi_{2x}(b_0,t)=0 & \\ \varphi_{2x}(x,0)=0 & (0\le x\le b_0) \\ \varphi_{2t}(b_0,t)=0. & \end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

which are definite problems of a series of linear equations. The solving method is analogeous to the case when $\gamma(\varphi_x)$ is a linear function. It's easy, so we will not describe it here.

References

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[3] A. Friedman: Partial Differential Equations of Parabolic Type, Prentic-Hall. Inc. (1964).