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Approximation by Integral Type Meyer-König and Zeller Operators*

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Abstract

This paper deals with the convergence rate and the L^p-saturation for approximation by integral type Meyer-König and Zeller operators. The open problems posed in [2] are solved.

1 The operators

$$M_n(f, x)=\sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{nk}(x), \quad 0\leq x<1,$$

$$m_{nk}(x)=\binom{n+k}{k} x^k(1-x)^{n+1}$$

are known as Meyer-König and Zeller operators. Their global behaviour was described by Becker and Nessel in [1]. However, we know that the operators M_n cannot be used in integral metrics. In order to extend M_n to L^p-metric Chen Wenzhong introduced in [2] the following modified operators

$$L_n(f,x)=\sum_{k=0}^{\infty}\left(C_{nk}^{-1}\int_0^1 m_{nk}(u)f(u)\mathrm{d}u\right)m_{nk}(x), \quad f\in L^1(0,1),$$

$$C_{nk}=\int_0^1 m_{nk}(u)\mathrm{d}u,$$

and called them the integral type Meyer-König and Zeller operators. As for approximation by L_n, Chen proved several results. In this paper, we discuss the L^p approximation by L_n. We have proved the quantitative theorem and L^p-saturation theorem for approximation by L_n. Therefore the open problems posed in [2] are solved.

Let $\varphi(x)=x(1-x)^2$, $1\leq p\leq\infty$, $\|\cdot\|_p=\|\cdot\|_{L^p(0,1)}$ and $S_p=\{f\in L^p(0,1);\ f'$ absolutely continuous, $\varphi f''\in L^p$ for $p>1$, or $\varphi f'\in BV$, $p=1\}$. The weighted modulus for $f\in L^p$ is defined by

$$\omega_\varphi(f,t)_p=\sup_{0<h\leq t}\|\triangle^2_{h\sqrt{\varphi}}f\|_p$$

$$\triangle^2_h f(x)=\begin{cases} f(x+h)-2f(x)+f(x-h), & x\pm h\in(0,1),\\ 0 & x\pm h\overline{\in}(0,1).\end{cases}$$

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Our main results are:

Theorem 1 For $f\in S_p$, we have

$$\|L_nf-f\|_p\leq\begin{cases} Cn^{-1}(\|f'\|_p+\|(\varphi f')'\|_p), & 1<p<\infty \quad (1.1)\\ Cn^{-1}(\|\varphi f'\|_\infty+\|\varphi f'\|_{BV}), & p=1. \quad (1.2)\end{cases}$$

where C is a constant depending only on p.

Theorem 2 Let $f\in L^p$. Then

$$\|L_nf-f\|_p\leq C(n^{-1}\|f\|_p+\omega_\varphi(f,n^{-\frac{1}{2}})_p). \quad (1.3)$$

Theorem 3 Suppose that $f\in L^p$. Then

(1) $\|L_nf-f\|_p=o(n^{-1})$ iff $f=$ const. a.e.;

(2) $\|L_nf-f\|_p=O(n^{-1})$ iff $f\in S_p$.

2 We now establish some lemmas

Lemma 1 Let $s(x)=x(1-x)^{-1}$. Then

$$L_n(s,x)-s(x)=(n+1)^{-1}.$$

Proof By calculating, we have

$$L_n(s,x)=\sum_{k=0}^{\infty}(C_{nk}^{-1}\int_0^1 t(1-t)^{-1}m_{nk}(t)\,dt)\,m_{nk}(x)$$

$$=\sum_{k=0}^{\infty}(\frac{(n+k+1)(n+k+2)}{n+1}\cdot\frac{(k+1)}{n}\int_0^1 m_{n-1,k+1}(t)\,dt)m_{nk}(x)$$

$$=\sum_{k=0}^{\infty}\frac{k+1}{n+1}m_{nk}(x)=(n+1)^{-1}+x(1-x)^{-1}.$$

The Lemma 1 is proved.

Lemma 2 Let $S_0=0$, $S_k=\sum_{i=1}^{k}i^{-1}$, $k\geq1$ and $r_{nk}=C_{nk}^{-1}\int_0^1\log t\,m_{nk}(t)\,dt$. Then

$$r_{nk}=r_{Ok+n}-(S_{n-k}-S_k),$$

$$r_{Ok+n}=-(n+k+1)^{-1}-(n+k+2)^{-1}.$$

Proof Integrating by parts we get

$$\int_0^1\log t\,m_{nk}(t)\,dt=-C_{nk}(k+1)^{-1}+n^{-1}(n+1)\int_0^1 m_{n-1,k+1}(t)\log t\,dt.$$

Thus

$$r_{nk}=-(k+1)^{-1}+C_{nk}^{-1}n^{-1}(n+1)\int_0^1 m_{n-1,k+1}(t)\log t\,dt$$

$$=-(k+1)^{-1}+C_{n-1,k+1}^{-1}\int_0^1 m_{n-1,k+1}(t)\log t\,dt=-(k+1)^{-1}-r_{n-1,k-1}.$$

Hence $r_{nk}=r_{Ok+n}-(S_{n+k}-S_k)$. On the other hand,

$$r_{Ok+n}=C_{Ok+n}^{-1}\int_0^1 m_{Ok+1}(t)\log t\,dt$$

$$=(n+k+1)(n+k+2)\int_0^1 t^{k+n}(1-t)\log t\,dt$$

$$=-(n+k+1)^{-1}-(n+k+2)^{-1}.$$

We complete the proof of Lemma 2.

Lemma 3 Let $r(x)=\log x-\log(1-x)$ and $p^{-1}+q^{-1}=1$ for $1\leq p<\infty$, or $q^{-1}=1$ for $p=\infty$. We have

$$\|\varphi^{\frac{1}{q}}(L_n r-r)\|_p=O(n^{-1}) \qquad (n\to\infty).$$

Proof Since L_n are linear operators, we need only prove the following two inequalities

$$\|\varphi^{\frac{1}{q}}(L_n(\log t,x)-\log x)\|_p\leq Cn^{-1} \tag{2.1}$$

and

$$\|\varphi^{\frac{1}{q}}(L_n(\log(1-t),x)-\log(1-x))\|_p\leq Cn^{-1}. \tag{2.2}$$

For $\sum_{k=0}^{\infty}(S_{n+k}-S_k)m_{nk}(x)=\sum_{i=1}^{n}i^{-1}(1-x)^i$, we have by lemma 2

$$L_n(\log t,x)-\log x=\sum_{k=0}^{\infty}\left(C_{nk}^{-1}\int_0^1\log t\,m_{nk}(t)\,dt\right)m_{nk}(x)+\sum_{k=1}^{\infty}k^{-1}(1-x)^k$$

$$=\sum_{k=0}^{\infty}(r_{Ok+n}-(S_{n+k}-S_k))m_{nk}(x)+\sum_{k=1}^{\infty}k^{-1}(1-x)^k$$

$$=O(n^{-1})+\sum_{k=n+1}^{\infty}k^{-1}(1-x)^k.$$

We here use the fact $\sum_{k=0}^{\infty}m_{nk}(x)r_{Ok+n}=O(n^{-1})$ (cf. [2, lemma 1]). Therefore

$$\int_0^1|L_n(\log t,x)-\log x|\,dx\leq Cn^{-1}+\sum_{k=n+1}^{\infty}(k(k+1))^{-1}\leq Cn^{-1}$$

and

$$\sup_{0\leq x\leq 1}|x(L_n(\log t,x)-\log x)|\leq\sup_{0\leq x\leq 1}\left(Cn^{-1}x+\sum_{k=n+1}^{\infty}k^{-1}x(1-x)^k\right)\leq Cn^{-1}.$$

By the Riesz-Thorin theorem (cf. [3, p.525]), we have proved the inequality (2.1).

With respect of (2.2), we have by calculating

$$L_n(\log(1-t),x)=\sum_{k=0}^{\infty}m_{nk}(x)(r_{Ok+n}-(S_{n+k}-S_{n+1}))$$

$$=O(n^{-1})-\sum_{k=0}^{\infty}m_{nk}(x)(S_{n+k}-S_{n+1})=O(n^{-1})+\log(1-x).$$

So $L_n(\log(1-t),x)-\log(1-x)=O(n^{-1})$, the inequality (2.2) holds apparently. The lemma 3 is proved.

3 The proof of theorem 1

Let $g(x)=s(x)+r(x)=\frac{x}{1-x}+\log\frac{x}{1-x}$. For any $x,t\in(0,1)$ and $f\in S_p$, there holds

$$f(t)-f(x)=\varphi(x)f'(x)(g(t)-g(x))+\int_t^x(g(u)-g(t))\,d(\varphi(u)f'(u)). \tag{3.1}$$

Applying the operators L_n to (3.1) in the variable t, we obtain

$$L_n(f,x)-f(x)=\varphi(x)f'(x)(L_n(g,x)-g(x))+L_n\left(\int_t^x(g(u)-g(t))\,d(\varphi(u)f'(u))\right).$$

Taking L^p-norms on both sides and applying lemmas 1 and 2, we obtain

$$\| L_n f - f \|_p \le C n^{-1} \| f' \|_p + \| L_n(\int_t^x (g(u) - g(t)) \mathrm{d}(\varphi(u) f'(u))) \|_p \quad (3.2)$$

(for $p=1$, $\| f' \|_p$ is replaced by $\| \varphi f' \|_\infty$).

In order to estimate the second term of the right in (3.2), we consider the linear operators

$$A_n(h, x) = n L_n(\int_t^x (g(u) - g(t)) \mathrm{d}h(u), x).$$

Write

$$(g(u) - g(t))_+ = \begin{cases} g(u) - g(t), & \text{if } u \ge t, \\ 0, & \text{if } u < t. \end{cases}$$

For $u \in BV$, we have

$$\int_0^1 |A_n(h, x)| \mathrm{d}x \le n \int_0^1 |\mathrm{d}h(u)| (\int_u^1 L_n((g(u) - g(\cdot))_+, x) \mathrm{d}x + \int_0^u L_n((g(\cdot) - g(u))_+, x) \mathrm{d}x).$$

Since $\int_0^1 (L_n(f, x) - f(x)) \mathrm{d}x = 0$ for any $f \in L^1$,

$$\int_u^1 L_n((g(u) - g(\cdot))_+, x) \mathrm{d}x + \int_0^u L_n((g(\cdot) - g(u))_+, x) \mathrm{d}x$$
$$= \int_0^u (L_n(g, x) - g(x)) \mathrm{d}x = O(n^{-1})$$

which implies

$$\int_0^1 | A_n(h, x) | \mathrm{d}x \le C \| h \|_{BV}.$$

Put $H_n(t, x) = \sum_{k=0}^{\infty} C_{nk}^{-1} m_{nk}(t) m_{nk}(x)$. Using lemmas 1 and 2, we obtain

$$|A_n(h, x)| = n L_n(\int_t^x (g(u) - g(t)) h'(u) \mathrm{d}u, x)$$
$$\le n \| h' \|_\infty \int_0^1 H_n(t, x) \int_t^x (g(u) - g(t)) \mathrm{d}u \mathrm{d}t$$
$$= n \| h' \|_\infty \int_0^1 H_n(t, x) (x(\log x - \log t) + (1 - x)(\log(1 - x)$$
$$- \log(1 - t)) + (\frac{t}{1 - t} - \frac{x}{1 - t}) + (\log(1 - t) - \log(1 - x))) \mathrm{d}t$$
$$= n \| h' \|_\infty \cdot O(n^{-1}) \le C \| h' \|_\infty.$$

The Riesz-Thorin theorem gives

$$\| A_n(h) \|_p \le C \| h' \|_p, \quad p \ge 1, \quad h' \in L^p(0, 1).$$

Taking $h(u) - \varphi(u) f'(u)$, we see that

$$\| L_n(\int_t^x (g(u) - g(t)) \mathrm{d}(\varphi(u) f'(u))) \|_p = n^{-1} \| A_n(\varphi f') \|_p$$
$$\le \begin{cases} C n^{-1} \| (\varphi f')' \|_p & p > 1, \\ C n^{-1} \| \varphi f' \|_{BV} \end{cases} \quad (3.3)$$

Combining (3.2) and (3.3), we complete the proof of theorem 1.

4 The proof of the theorem 2

For any $h'' \in L^p(0, 1)$, from theorem 1 we can also obtain

$$\| L_n h - h \|_p \le C n^{-1} (\| h \|_p + \| \varphi h'' \|_p).$$

For $f \in L^p(0, 1)$, we have (restricting $\| h \|_p \le 2 \| f \|_p$)

$$\|L_nf-f\|_p\le\|L_n(f-h)\|_p+\|f-h\|_p+\|L_nh-h\|_p$$
$$\le 2\|f-h\|_p+Cn^{-1}(\|\varphi h''\|_p+\|h\|_p)$$
$$\le C(\|f-h\|_p+n^{-1}\|\varphi h''\|_p)+Cn^{-1}\|f\|_p.$$

This implies

$$\|L_nf-f\|_p\le CK_p(f,n^{-\frac{1}{2}})+Cn^{-1}\|f\|_p$$

where

$$K_p(f,t)=\inf\{\|f-h\|_p+t\|\varphi h''\|_p,\ \ \varphi h''\in L^p(0,1)\}.$$

Applying the weak equivalent relationship between $K_p(f,t^2)$ and $\omega_\varphi(f,t)_p$ (cf. [4, Theorem 2.1.1]), we have

$$\|L_nf-f\|_p\le C(n^{-1}\|f\|_p+\omega_\varphi(f,n^{-\frac{1}{2}})_p)$$

which provides the theorem 2.

5 The proof of the theorem 3

The theorem 3 follows from the theorem 1 above and the theorems 5 and 6 in [2].

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积分型Meyer-König-Zeller算子的逼近性质

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摘　　要

本文讨论了积分型Meyer-König-Zeller算子的逼近度和饱和性质。所得结论表明，积分型Meyer-König-Zeller算子和Kantorovich型Meyer-König-Zeller算子有相同的逼近阶、饱和阶及饱和类。