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The Iterative Roots for a Class of Self Mapping on S^1 *

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Abstract

The paper gives a necessary and sufficient condition that the sectionally strictly monotone continuous mapping that $\deg(f)=0$ on S^1 exists every order iterative roots, and gets that f exists every order iterative roots is equivalent to can be embedded in a quasi-semiflow.

Let E be a set, and $f, g: E\to E$ be mappings. We say that g is the n order iterative root of f if the equation $g^n = f$ hold for every $x\in E$. Given a set E, f: $E\to E$ and a positive integer n, can we find $g: E\to E$ such that $g^n = f$? This is the problem of the existance of iterative root. From end of the 19th Century one began to research it. So far, many papers have be published. In particular, [1], [2] obtain very well results.

It is known, for a continuous self mapping f on a topological space, that f has every order continuous iterative roots if f can be embedded in a quasi-semiflow. It is natural to ask that if f has every order iterative roots, can f be embedded in a quasi-semiflow? The paper studies the two problems above for sectionally strictly monotone continuous mappings that $\deg(f)=0$ on S^1. We obtain the similar results to [1] and show that existing every order iterative roots is equivalent to can be embedded in a quasi-semiflow. From these results we further recognize that there are close connections between the research for interval and for S^1.

Let R be the real line and $S^1=\{z\in C\,|\,|z|=1\}$ be the unit circle. We define the positive direction of S^1 as the counterclockwise. $p: R\to S^1$ defined by $p(x)=e^{2\pi i x}$ is a covering mapping. From the theory of algebraic topology, there exists a continuous mapping $F: R\to R$ for every continuous mapping $f: S^1\to S^1$ such that the figure above can be exchanged. F is called a lifting of f.

$$\begin{tikzcd} R \arrow[r, "F"] \arrow[d, "p"'] & R \arrow[d, "p"] \\ S^1 \arrow[r, "f"] & S^1 \end{tikzcd}$$

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In what follows, f, g indicate continuous self mappings on S^1. F(or F') and G (or G') represent the lifting of f and g respectively. For $[A, B] \subset R$, $F_{A,B}$ denote restricfion of $F: R \to R$ on $[A, B]$.

Lemma 1 If F is a lifting of f. Then $F+k$ is also a lifting of f for every integer $k \in Z$.

Proof It is easy to prove from the definition.

Lemma 2 If both F and F' are liftings of f, then there exists an integer $k \in Z$ such that $F = F' + k$.

Proof Since $p \circ F = f \circ P = P \circ F'$, thus there exist integers $k(x) \in Z$ for $x \in R$ such that $F(x) = F'(x) + k(x)$. Let $(F - F')(x) = F(x) - F'(x)$. Then $F - F'$ is a continuous mapping. Because $(F - F')(\mathrm{R})$ is a connected subset of R, and $(F - F')(x) = k(x) \in Z$, hence there exists an integer k jndependenting of x such that $(F - F')(x) = k$ (for every $x \in R$), i.e., $F = F' + k$,

Lemma 3 If $g^n = f$, then $[\deg(g)]^n = \deg(f)$.

Proof If is easy to prove from the theory of algebraic topology.

From Lemma 3 we see that the necessary condition that f exists every order iterative roots is $\deg(f) = 0$ or 1.

Lemma 4 If $[\deg(g)]^n = \deg(f)$ for n. then $g^n = f$ if and only if for every G there exists F such that $G^n = F$.

Proof Let $G^n = F$. Because $p \circ G = g \circ p$, $p \circ F = f \circ p$, hence $f \circ p = p \circ G^n = g^n \circ p$. Since $p: R \to S^1$ is a surjection, thus $g^n = f$.

Conversely, let $g^n = f$ and F_1 be a lifting of f. Since $p \circ G = g \circ p$, $p \circ F_1 = f \circ p$, so $p \circ G^n = p \circ F_1$. From Lemma 2, there exists a integer $k \in Z$ such that $G^n = F_1 + k$. Let $F = F_1 + k$. Using Lemma 1, F is also a lifting of f and $G^n = F$.

Lemma 5 Let $\varphi, \psi: R \to R$ be continuous mappings and $\psi^n = \varphi$. Let φ', ψ' be the represensations of φ, ψ under the transformation of coordinates. $x' = x + a$, $y' = y + a$, respectively. Then $(\psi')^n = \varphi'$.

Proof Since $\varphi(x) = \varphi'(x') - a$, $\psi'(x') = \psi'(x + a) = \psi(x) + a$, thus $(\psi')^2(x') = \psi' \psi'(\psi(x) + a) = \psi^2(x) + a$. Similarly, then $(\psi')^n(x') = \psi^n(x + a) = \varphi(x) + a = \varphi'(x')$.

Lemma 5 shows that the relation of f and its iterative root g is independent of which point is taken to act as the origin.

In what follows, we suppose $\deg(f) = 0$. It is easy to see that the lifting $F: R \to R$ of f is a continuous periodic function of period 1. Thus F has maximum and minimum. Denote $M = \max(F)$, $m = \min(F)$. We can suppose that f isn't a constant mapping. Thus $M > m$.

Lemma 6 If g is a n order iterative root of f. Then for every F there is G such that $G^n = F$. If $M - m \leq 1$, then $\max(G) - \min(G) \leq 1$ and $G(R) \supset F(R)$; If $M - m > 1$, then $G(R) = F(R)$.

Proof By Lemma 4, for every G' there is F' such that $(G')^n = F'$. Since $\deg(f) = \deg(g) = 0$, thus $G'(x+k) = G'(x)$, $F'(x+k) = F'(x)$ for every integer $k \in Z$, From Lemma 2, there is an integer $k \in Z$ for F, F' such that $F = F' + k$. Now let $G = G' + k$, then $G^n = (G')^n + k = F' + k = F$.

When $M - m \leq 1$. Since $G^n = F$, hence $\min(G) \leqq m < M \leq \max(G)$, and $G(R) \supseteq F(R)$. Now we are ready to prove that $\max(G) - \min(G) \leq 1$, Assume this is not true. Because the period of g is 1, $G^2([0,1]) = G([\min(G), \max(G)]) \supseteq G([0,1]) \supseteq [\min(G), \max(G)]$. Then $M - m = \max(G^n) - \min(G^n) \geqq \max(G) - \min(G) > 1$. This contradicts $M - m \leqq 1$, so $\max(G) - \min(G) \leqq 1$.

When $M - m > 1$. Let $D = M - m$. Then $F(R) \subset [F(0) - D, F(0) + D]$. By $G^n = F$, hence $[m, M] \subseteq [\min(G), \max(G)]$, $\max(G) - \min(G) \geq M - m = D > 1$. Thus there is $L \in R$ such that $[L, L+1] \subset [\min(G), \max(G)] = G([0,1])$, and $[\min(G), \max(G)] = G([L, L+1]) \subseteq G^2([0,1])$. We have further $[\min(G), \max(G)] \subseteq G^i([0,1])$ for $i \geq 1$. In particular, $[\min(G), \max(G)] \subseteq G^n([0,1]) = F([0,1]) = [m, M]$. Thus $[\min(G)$, ma $\max(G)] = [m, M]$, i.e., $F(R) = G(R)$.

Definition | Let $f: S^1 \to S^1$ be a continuous mapping on S^1 If the restriction $F_{0,1}$ on $[0,1]$ of the lifting F: $R \to R$ of f is sectionally strictly monotone, then f is called the sectionally strictly monotone mapping. If $\xi \in R$ is a extremal point of F, then $p(\xi) \in S^1$ is called the extremal point of f. The maximum and minimum can be definod similarly. $N(f)$ indicate the number of extremal points of f. Denote $N(F) = N(f)$. $N(F)$ is called the number of extermal points of F (under the mod1 equivalence meaning). It is easy to see that for a lifting F of f, F^m is a lifting of f^m for cvery $m \in Z^+$. Thus $N(F^m) = N(f^m)$. From [1] we have $0 = N(f^0) \leq N(f^1) \leq N(f^2) \leq \cdots \leq N(f^m) \leq \cdots\cdots$

When there exists m such that $N(f^m) = N(f^{m+1})$, then let $H(f) = \min\{m | N(f^m) = N(f^{m+1})\}$.

In the following, the mapping $f: S^1 \to S^1$ satisfies $\deg(f) = 0$. First we have a result as follows:

Proposition | Let f be a sectionally strictly monotone continuous mapping. If $H(f) > 1$, $n > N(f)$, then f hasn't n order iterative roots.

Proof Let g be a n order iterative root of f. Using Lemma 6, there is G for every F such that $G^n = F$.

If $M - m \leq 1$, then Lemma 6 shows that $\max(G) - \min(G) \leq 1$ and $G(R) \supseteq F(R)$. From Lemma 3, the proper coordinate system can be selected such that $F(R) \subseteq G(R) \subseteq [0,1]$. Thus $G_{0,1}$ and $F_{0,1}$ are mappings from $[0,1]$ to $[0,1]$, and $G_{0,1}^n = F_{0,1}$. This contradicts Theorem 1 of [1].

If $M - m > 1$, then Lemma 6 shows that $F(R) = G(R)$. By Lemma 3 we can assume $m = 0$. So $F(R) = G(R) = [0, M]$. Let $K = \min\{x \in Z^+ | x \geq M\}$. Then $G_{0,k}$ and

$F_{0,k}$ are mappings from $[0,k]$ to $[0,k]$ and $G^n_{0,k}=F_{0,k}$. Since $H(f)>1$, then $H(F_{0,k})>1$. According to the definition of $H(\cdot)$, we obtain $N(f^2)>N(f)$. Thus $N(g^{2n})>N(g^n)$, $N(G^{2n}_{0,k})>N(G^n_{0,k})$. From Lemma 8 of [1], we have $H(G_{0,k})>n$. By the periods of G and F are 1, hence

$$0\leq N(G^0_{0,k})<N(G^1_{0,k})<\cdots<N(G^n_{0,k})=N(F_{0,k})=KN(F)$$

On the other hand, from $N(G^{i+1}_{0,k})>N(G^i_{0,k})\ (0\leq i\leq n-1)$ and the period property of G, then $N(G^{i+1}_{0,k})-N(G^i_{0,k})>K\ (0\leq i\leq n-1)$. Thus $N(F_{0,k})=N(G^n_{0,k})>nk>KN(F)$. This is a contradiction and finishes the proof of the proposition.

Proposition 1 shows that only $H(f)=1$ can f have every order iterative roots.

Lemma 7 If $\varphi:[a,b]\to[a,b]$ is a continuous mapping, then $R=R_1\cup R_2$. Where R and R_1 denote the nonmonotone points sets of φ^2 and φ respecfively, and $R_2=\{x|\varphi(x)\in R_1\}$.

Proof See the Lemma 3′ of [1].

Proposition 2 If $\deg(f)=0$, then $H(f)=1$ if and only if there exist $a,b\in S^1$ (to assume $a<b$) such that f is strictly monotone on $[a,b]\subset S^1$, and $f(S^1)=[a,b]$.

Proof Sufficiency. Let $A,B\in R$ and $0\leq A<B\leq 1$ such that $p(A)=a$, $p(B)=b$. Then the lifting F of f is strictly monotone on $[A,B]$. Since $f(S^1)=[a,b]$, we can assume $F([0,1])=[A,B]$. This is $0\leq m=A<B=M\leq 1$. It follows that $H(F_{0,1})=1$ from Lemma 7 of [1]. Hence $H(F)=1$, $H(f)=1$.

Necessity. First we prove $M-m\leq 1$. If f is constant, the conclusion is evident. If f isn′t constant, because $\deg(f)=0$, hence $N(f)\geq 2$, i.e., $N(F)\geq 2$. Assume $M-m>1$. Set $p(0)\in S^1$ is a monotone point of f. Thus $(0,1)$ contains the extremal points set of f. We can take $A\in R$ such that $m<A<A+1<M$. If $F(x_0)=m$, $F(y_0)=M$ for $x_0,y_0\in(0,1)$, then there exist t_0,S_0 between x_0 and y_0 such that $F(t_0)=A$, $F(S_0)=A+1$,

When $S_0>t_0\ (t_0>S_0)$, then $F([t_0,S_0])\supseteq[0,1]\ (F([S_0,t_0])\supseteq[0,1])$. Hence we can take $K\in Z^+$ such that $-K\leq m<M\leq K$. So $F_{-k,k}$ is a mapping from $[-k,k]$ to $[-k,k]$. From Lemma 7 we know that $N(F^2)\geq N(F)+2>N(F)$. This contradicts $H(f)=1$. Thus $M-m\leq 1$.

Assume $m=0$. Then $F_{0,1}$ is a mapping from $[0,1]$ to $[0,1]$. Set $A=0=\min(F)$, $B=M=\max(F)$. From Lemma 7 of [1], $F_{0,1}$ is strictly monotone and so is F on $[A,B]$. $F([0,1])=[A,B]$. Let $a=p(A)$, $b=p(B)$. Then the necessity has be proved.

According to Proposition 2 and using the same method as [1], we can give the concept called "characteristic interval" as follows:

Definition 2 If f exists extremal points $a,b\in S^1$ $(a<b)$ such that $f(S^1)\subseteq[a,$

$b]$ and f is strictly monotone on $[a, b] \subset S^1$, then $[a, b]$ is called a "characteristic interval" of f.

From Proposition 2, there is a lifting F of f such that $F_{0,1}$ is a mapping from $[0,1]$ to $[0,1]$ and has the characteristic interval $[A, B]$. Moreover, it is easy to see for $\deg(f)=0$ that f has n order iterative roots iff every lifting F has n order iterative roots. Therefore, if $F_{0,1}$ has the n order iterative root G on $[0, 1]$ such that $G(0)=G(1)$, then we can extend G by period 1 to $\boldsymbol{R}$. It is easy to see that there is an induced by G continuous mapping g on S^1 such that $p \circ G = g \circ p$. This g is obviously the n order iterative root of f. Thus, to seek the n order iterative root of f, it is enough to seek the n order iterative root G of F satisfying $G(0)=G(1)$. However, according to Theorem 4 of [1] and its proof, to obtain the iterative roof of F, we can seek the iterative root satisiying some conditions of the restriction $F_{A,B}$ of F on its characteristic interval $[A, B]$, and then extend it to an iterative root of F. Since $F(0)=F(1)$, it is follow that there is the iterative root G of F such that $G(0)=G(1)$ from the method of proving therem 4 in [1]. Then as stated above, we can get the iterative root of f. Hence, for the iterative root of f with $\deg(f)=0$ and $H(f)=1$, we can write all conclusions corresponding to the existence theorems of the iterative root in [1]. Of course some of these conclusions can be got directly by using the conclusion on embedded in quasi-semiflows in [3] (for example sufficiency of Theorem 2). Now we state the main conclusions as follows:

Theorem 1 If $\deg(f)=0$, f exists the characteristic interval $[a, b] \subset S^1$ and f is strictly decreasing on $[a, b]$, then

(a) If n is an even number, then f doesn't exist n order iterative roots.

(b) If n is an odd number and that $f(a)=b$, $f(b)=a$ or that $f(a)<b$, $f(b)>a$ hold at the same time, then f exists n order iterative roots.

Theorem 2 If $\deg(f)=0$, then f exists every n order iterative roots if and only if f exists a characteristic interval $[a, b] \subset S^1$ such that f is strictly imcreasing on $[a, b]$.

Finally, combining Theorem 2 with the results in [3] we obtain.

Theorem 3 If $\deg(f)=0$, then that f exists every n order iterative roots is equivalent to can be embedded in a quasi-semiflow.

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S^1 上一类自映射的迭代根

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摘　　要

本文给出了圆周上 dege (f) = 0 的分段严格单调的连续映射 f 存在任意阶迭代根的充分必要条件，得出了 f 存在任意阶迭代根等价于可以嵌入一个拟半流的结论.