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Time Behaviour of Solution of the Porous Medium Equation with Absorption*

Yang Jinshun
(Northeastern Electric Power Institute, Jilin, China)

Zhao Junning
(Dept. of Math., Jilin University, Changchun, China)

1. Introduction

We consider the Cauchy problem

$$u_t = \Delta u^m - f(u), \quad \text{in} \quad S = R^N \times (0,\infty), \tag{1}$$
$$u(x,0) = \varphi(x), \quad \text{in} \quad R^N, \tag{2}$$

where ϕ is a given bounded nonnegative function and f is a C^1 function such that

$$f(0) = 0 \quad \text{and} \quad f(u) > 0 \quad \text{if } u > 0.$$

Equation (1) has been suggested as a mathematical model for a variety of physical problems. We shall not recall them here, but refer to [1], where the very extensive literature about the porous medium equation and some of its generalizations is summarized.

The existence and uniqueness of a nonnegative solution of (1) and (2) defined in some weak sense, are well established in [2], [3].

In this paper, we are interested in the behaviour of solution as $t \to \infty$. Suppose that

$$\text{H1:} \quad \lim_{|x|\to\infty} |x|^\alpha \varphi(x) = A,$$
$$\text{H2:} \quad \lim_{s\to 0} s^{-p} f(s) = \sigma,$$

where σ is a positive constant. In [4], it is shown that

1. If $p > m \geq 1$ and $0 < \alpha < \frac{2}{p-m}$, then

$$t^{\frac{1}{p-1}} u(x,t) \to C^* \sigma^{-\frac{1}{p-1}}$$

uniformly on sets of the form

$$\{x \in R^N : |x| \leq a t^{\frac{1}{\beta}}\}, a \geq 0, t > 0,$$

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where $\beta = \frac{2(p-1)}{p-m}$ and $C^* = (p-1)^{\frac{1}{p-1}}$.

2. If $p > m + \frac{2}{N}, m > 1, \frac{2}{p-m} < \alpha < N$ and for every fixed $\omega \in R^N, |\omega| = 1, \varphi$ satisfies

$$|x|^\alpha |\varphi(x)| \leq B, \quad \text{for all } x \in R^N,$$

$$\lim_{x \to \infty} |x|^\alpha \varphi(|x|\omega) = A(\omega),$$

in which $A(\omega) \geq 0$ ($\not\equiv 0$). Then

$$t^{\frac{\alpha}{\gamma_0}} u(x,t) \to h(xt^{-\frac{1}{\gamma_0}}) \quad \text{as} \quad t \to \infty,$$

uniformly on sets of the form $\{x \in R^N; |x| \leq at^{\frac{1}{\gamma}}, \ a \geq 0\}$. Here $h(\xi)$ is a positive solution of the problem

$$\Delta h^m + \frac{1}{\gamma}\xi \cdot \nabla h + \frac{\alpha}{\gamma} h = 0, \quad \xi \in R^N,$$

$$\lim_{|\xi| \to \infty} |\xi|^\alpha h(|\xi|\omega) = A(\omega).$$

3. If $p > m + \frac{2}{N}, m > 1, \alpha > N$, then

$$t^{\frac{1}{\delta}} |u(x,t) - E_{C_0}(x,t)| \to 0, \quad \text{as} \quad t \to \infty,$$

uniformly on sets of the form $\{x \in R^N; |x| \leq at^{\frac{1}{N\delta}}\}$, $a \geq 0$, where $\delta = m - 1 + \frac{2}{N}$, E_{C_0} is the Barenblatt–Pattle solution with mass C_0 and

$$C_0 = |\varphi|_{L^1} - \int_0^\infty \int_{R^N} f(u) dx\, dt.$$

The authors of [4] conjecture that if $m < p < m + \frac{2}{N}, m > 1, \alpha > \frac{2}{p-m}, f(u) = u^p$, then

$$t^{\frac{1}{p-1}} |u(x,t) - U(x,t)| \to 0 \quad \text{as } t \to \infty$$

uniformly on sets of the form $\{x \in R^N; |x| < at^{\frac{1}{\beta}}\}, a \geq 0$, where $\beta = 2(p-1)/(p-m)$, and $U(x,t)$ is the very singular solution of (1) (see [5]), i.e., a solution with the properties

$$U_t = \Delta U^m - U^p \qquad \text{in } D'(s), \tag{3}$$

$$U \in C(\bar{S} \setminus \{(0,0)\}), \qquad U(x,0) = 0 \ \text{ if } x \neq 0 \tag{4}$$

$$\lim_{t \to 0} \int_{|x|<R} U(x,t) dx = +\infty \quad \text{for every} \quad R > 0. \tag{5}$$

In this paper we give the proof of the conjecture in [4] for $m > (1 - \frac{2}{N})^+$. As a corollary, the existence of very singular solution of (3) is proved, which is different to the method in [5].

2. The Proof of Theorem

Definition A nonnegative function $u \in L^\infty(S)$ is called a generalized solution of (1), (2), if u satisfies

$$\int\int_S \{u\xi_t + u^m\Delta\xi - f(u)\xi\}dx\,dt + \int_{R^N} \varphi(x)\xi(x,0)dx = 0, \tag{6}$$

for any $\xi \in C^{2,1}(\bar{S})$ which vanishes for large $|x|$ and t.

Theorem 1 *Suppose that* $\max\{1,m\} < p < m + \frac{2}{N}, m > (1-\frac{2}{N})^+, \alpha > \frac{2}{p-m}$ *and* φ, f *satisfy H1, H2. Then*

$$t^{\frac{1}{p-1}} \mid u(x,t) - U(x,t) \mid \to 0 \quad as \quad t \to \infty$$

uniformly on sets of the form $\{x \in R^N; |x| < at^{\frac{1}{\beta}}\}, a \geq 0$, *where* $\beta = 2(p-1)/(p-m)$, *and* U *is the very singular solution of equation*

$$U_t = \Delta U^m - \sigma U^p. \tag{7}$$

We begin with some preliminary discussions.

Let u be the solution of (1) and (2). We consider the family of functions

$$u_k = k^{\frac{2}{p-m}} u(kx, k^\beta t).$$

It is a solution of the following problem

$$u_t = \Delta u^m - k^{\frac{2p}{p-m}} f(k^{-\frac{2}{p-m}} u) \text{ in } S, \tag{8}$$
$$u(x,0) = k^{\frac{2}{p-m}} \phi(kx) \text{ on } \mathbf{R}^N. \tag{9}$$

Below we shall denote by C the constants independent of k, although they may change from line to line in the proof. Denote

$$\begin{aligned} B_R(x_0) &= \{x; |x - x_0| < \mathbf{R}\}, \\ Q_R(x_0,t_0) &= B_R(x_0) \times (t_0 - \mathbf{R}^2, t_0), \\ S_T &= R^N \times (0,T). \end{aligned}$$

In the following Lemmas, we assume $u_k \in C^2(\bar{S})$, otherwise, by the uniqueness of solution of (8), (9) [6], we can consider the approximate problem, as in [7].

Lemma 1 *For every* $r, t_1, k > 0, u_k$ *satisfies*

$$\int_{t_1}^T \int_{B_R(x_0)} u_k^r dx\,dt \leq C.$$

Proof Let $\xi \in C_0^\infty(S), 0 \le \xi \le 1, \xi = 1$ on $B_R(x_0) \times (t_1, T)$. We multiply (8) by $u_k^{m\alpha}\xi^2$ and integrate over S_T to obtain

$$\int_{R^N}\int_0^{u_k(x,T)} s^{m\alpha}\xi^2 ds\, dx + \int_0^T\int_{R^N} k^{\frac{2p}{p-m}} f(k^{-\frac{2}{p-m}}u_k)(u_k)^{m\alpha}\xi^2 dx\, dt$$
$$\le \int_0^T\int_{R^N}\left[|\nabla\xi|^2(u_k)^{m\alpha+m} + |\xi_t|\,\xi\int_0^{u_k} s^{m\alpha}ds\right]dx\, dt. \tag{10}$$

Note that, by H2, there exists a constant d such that if

$$k^{-\frac{2}{p-m}}u_k = u(kx, k^\beta t) \le d,$$

then

$$k^{\frac{2p}{p-m}} f(k^{-\frac{2}{p-m}}u_k) \ge \frac{1}{2}\sigma u_k^p.$$

Moreover, if

$$u(kx, k^\beta t) \ge d,$$

we have also

$$k^{\frac{2p}{p-m}} f(k^{-\frac{2}{p-m}}u_k) \ge C(d)k^{\frac{2p}{p-m}}u^p(kx, k^\beta t) = C(d)u_k^p(x,t).$$

Thus we have from (10)

$$\int_0^T\int_{R^N}(u_k)^{m\alpha+p}\xi^2 dx\, dt \le C\{\int_0^T\int_{R^N}|\nabla\xi|^2(u_k)^{m\alpha+m}dx\, dt$$
$$+ \int_0^T\int_{R^N}|\xi_t|\,\xi(u_k)^{m\alpha+1}dx\, dt\}. \tag{11}$$

We choose $\xi = \psi^r$ in (12), where $r = (\alpha m + p)/(\alpha m + p - \max\{m\alpha + m, m\alpha + 1\}), 0 \le \psi \le 1, \psi \in C_0^\infty(S), \psi = 1$ on $B_R(x_0) \times (t_1, T)$, to obtain

$$\int_0^T\int_{R^N}\psi^{2r}(u_k)^{m\alpha+p}dx\, dt$$
$$\le C_1[\int_0^T\int_{R^N}\psi^{2r}(u_k)^{m\alpha+p}dx\, dt]^{\frac{m\alpha+m}{m\alpha+p}} + C_1[\int_0^t\int_{R^N}\psi 2r(u_k)^{m\alpha+p}dx\, dt]^{\frac{m\alpha+1}{m\alpha+p}}$$
$$\le C_2 + C_2\{\int_0^T\int_{R^N}\psi^{2r}(u_k)^{m\alpha+p}dx\, dt\}^{\frac{\max\{m\alpha+m, m\alpha+1\}}{m\alpha+p}}.$$

This implies

$$\int_0^T\int_{R^N}\psi^{2r}(u_k)^{m\alpha+p}dx\, dt \le C.$$

Thus Lemma 1 is proved. □

Lemma 2 *Let $\overline{B_R(x_0)} \subset R^N \setminus \{0\}$. Then u_k satisfies*

$$\int_0^T\int_{B_R}(u_k)^r dx\, dt \le C \qquad \text{for every } r < 0.$$

Proof Let $\xi(x) \in C_0^\infty(R^N)$, $0 \le \xi \le 1, \xi = 1$ if $x \in B_R(x_0)$, $\mathrm{supp}\xi \subset R^N \setminus \{0\}$. We multiply (8) by $(u_k)^{m\alpha}\xi^2$ and integrate over $R^N \times (0,T)$ to obtain

$$\int_{R^N}\int_0^{u_k(x,t)} s^{m\alpha}\xi^2\,ds\,dx + \int_0^T\int_{R^N} k^{\frac{2p}{p-m}} f(k^{-\frac{2}{p-m}}u_k)(u_k)^{m\alpha}\xi^2\,dx\,dt$$
$$\le C\left[\int_0^T\int_{R^N} |\nabla\xi|^2(u_k)^{m\alpha+m}\,dx\,dt + \int_{R^N}\xi^2\int_0^{k^{\frac{2}{p-m}}\varphi(kx)} S^{m\alpha}\,ds\,dx\right].$$

Note that by H1 if k is large enough,

$$k^{\frac{2}{p-m}}\varphi(kx) \le C \qquad \text{on } \mathrm{supp}\xi.$$

Thus

$$\int_{R^N}\xi^2\int_0^{k^{\frac{2}{p-m}}\varphi(kx)} s^{m\alpha}\,ds\,dx$$

is uniformly bounded for k. Hence using the analogous argument to Lemma 1, we can prove Lemma 2.

Lemma 3 *Let $t_1 > 0$. Then*

$$\sup_{R^N\times(t_1,T)} u_k \le C(t_1). \tag{12}$$

Proof Let $\xi \in C_0^\infty(B_R \times (0,T)), 0 \le \xi \le 1$. We multiply (8) by $\xi^2(u_k)^{m(2r-1)}$ with $r > 1$ and integrate over $B_R \times (t_0 - R^2, t)$ to obtain

$$\int_{B_R}\xi^2(u_k(x,t))^{m(2r-1)+1}dx + \int_{t_0-R^2}^t\int_{B_R}\xi^2|\nabla u_k^{mr}|^2dx\,ds$$
$$\le C\{\int_{t_0-R^2}^t\int_{B_R}|\nabla\xi|^2u_k^{2mr}dx\,ds + \int_{t_0-R^2}^t\int_{B_R}|\xi_t|\,u_k^{m(2r-1)+1}dx\,ds\},$$

where C does not depend on r.

Hence we have

$$\sup_{t\in t_0-R^2,t_0}\int_{B_R}\xi^2u_k^{m(2r-1)+1}dx + \int\int_{Q_R}|\nabla(\xi u_k^m)^r|^2dx\,dt$$
$$\le C\{\int\int_{Q_R}|\nabla\xi|^2u_k^{2mr}dx\,dt + \int\int_{Q_R}|\xi_t|\,u_k^{m(2r-1)+1}dx\,dt\}. \tag{13}$$

Using the embedding inequality [8, p.62, p.74], we have

$$\begin{aligned}
&\iint_{Q_R} (\xi u_k^m)^{2r+4\frac{r}{N}-\frac{2}{N}+\frac{2}{mN}}\,dx\,dt \\
\le\ & C\left\{\sup_{t_0-R^2<t<t_0}\int_{B_R}(\xi u_k^m)^{2r-1+\frac{1}{m}}dx\right\}^{\frac{2}{N}}\cdot\iint_{Q_R}|\nabla(\xi u_k^m)^r|^2dx\,dt \\
\le\ & C_1\left\{\sup_{t_0-R^2<t<t_0}\int_{B_R}(\xi u_k^m)^{2r-1+\frac{1}{m}}dx+\iint_{Q_R}|\nabla(\xi u_k^m)^r|^2dx\,dt\right\}^{1+\frac{2}{N}} \\
\le\ & C_2\left\{\iint_{Q_R}|\nabla\xi|^2u_k^{2mr}dx\,dt+\iint_{Q_R}|\xi_t|\,u_k^{m(2r-1)+1}dx\,dt\right\}^{1+\frac{2}{N}}. \qquad (14)
\end{aligned}$$

We first consider the case when $m<1$. Let

$$R_j=R(\frac{1}{2}+\frac{1}{2^{j+1}}),\quad j=1,2,\cdots$$

and let $\xi_j\in C_0^2(B_{R_j}\times(t_0-R^2,T))$ with $\xi_j=1$ in $Q_{R_{j+1}}$. Denote $1+\frac{2}{N}$ by K and take r such that

$$2r=(\frac{1}{m}-1)(\frac{N}{2}-1)+K^j,\quad j=j_0,j_0+1,j_0+2,\cdots,$$

where j_0 is a natural number such that

$$(\frac{1}{m}-1)(\frac{N}{2}-1)+K^j>2.$$

From (14) we get

$$\begin{aligned}
&\iint_{Q_{R_{j+1}}}(u_k^m)^{\frac{N}{2}(\frac{1}{m}-1)+K^{j+1}}dx\,dt \\
\le\ & \{\frac{C4^j}{R^2}[\iint_{Q_{R_j}}(u_k^m)^{\frac{N}{2}(\frac{1}{m}-1)+K^j}dx\,dt \\
+\ & \iint_{Q_{R_j}}(u_k^m)^{\frac{N}{2}(\frac{1}{m}-1)+1-\frac{1}{m}+K^j}dx\,dt]\,\}^K. \qquad (15)
\end{aligned}$$

Let $u_{1k}=\max\{1,u_k\}$. From (15) we obtain

$$\begin{aligned}
&\iint_{Q_{R_{j+1}}}(u_{1k}^m)^{\frac{N}{2}(\frac{1}{m}-1)+K^{j+1}}dx\,dt \\
\le\ & \iint_{Q_{R_{j+1}}}(u_k^m)^{\frac{N}{2}(\frac{1}{m}-1)+K^{j+1}}dx\,dt+mes\,Q_R \\
\le\ & \{C(R)4^j\iint_{Q_{R_j}}(u_{1k}^m)^{\frac{N}{2}(\frac{1}{m}-1)+K^{j+1}}dx\,dt\}^K.
\end{aligned}$$

The standard Moser's iteration yields

$$\sup_{Q_{\frac{R}{2}}}u_{1k}^m\le C(R)\{\iint_{Q_R}(u_{1k}^m)^{\frac{N}{2}(\frac{1}{m}-1)+K^{j_0}}dx\,dt\}^{\frac{1}{K^{j_0}}}.$$

Hence by Lemma 1, the Lemma 3 when $m < 1$ is proved.

If $m \geq 1$, let $2r = 1 - \frac{1}{m} + K^j$ in (14) to obtain

$$\begin{aligned}&\iint_{Q_{R_{j+1}}} (u_k^m)^{1-\frac{1}{m}+K^{j+1}} dx\,dt \\ \leq\ & \{C\frac{4^j}{R^2}\left[\iint_{Q_{R_j}} (u_k^m)^{1-\frac{1}{m}+K^j} dx\,dt + \iint_{Q_{R_j}} (u_k^m)^{K^j}\right]\}^K.\end{aligned}$$

Thus

$$\iint_{Q_{R_{j+1}}} (u_{1k}^m)^{1-\frac{1}{m}+K^{j+1}} dx\,dt \leq \{C\frac{4^j}{R^2}\iint_{Q_{R_j}} (u_{1k}^m)^{1-\frac{1}{m}+K^j} dx\,dt\}^K.$$

The standard Moser's iteration yields

$$\sup_{Q_{\frac{R}{2}}} u_{lk}^m \leq C(R)\{\iint_{Q_R} (u_{lk}^m)^{1-\frac{1}{m}+K} dx\,dt\}^{\frac{1}{k}}.$$

Hence by Lemma 1, the Lemma 3 is proved.

Lemma 4 *Let* $\overline{B_R(x_0)} \subset R^N \setminus \{0\}$. ***Then***

$$\sup_{B_R(x_0)\times(0,T)} u_k \leq C. \tag{16}$$

Proof Let $\xi(x) \in C_0^2(R^N), 0 \leq \xi \leq 1, \operatorname{supp}\xi \subset R^N \setminus \{0\}, \xi = 1$ on $B_R(x_0)$. We multiply (8) by $\xi^2 u_k^{m(2r-1)}$ with $r > 1$ to obtain

$$\begin{aligned}&\int_{R^N} \xi^2 u_k^{m(2r-1)+1}(x,t)\,dx + \int_0^t\int_{R^N} \xi^2|\nabla u_k^{mr}|^2 dx\,ds \\ \leq\ & C\{\int_0^t\int_{R^N} |\nabla\xi|^2 u_k^{2mr} dx\,ds + \int_{R^N} \xi^2(k^{\frac{2}{p-m}}\varphi(kx))^{m(2r-1)+1}\}.\end{aligned} \tag{17}$$

Since $\operatorname{supp}\xi \subset R^N \times \{0\}$, by H1,if k is large enough, we have

$$k^{\frac{2}{p-m}}\varphi(kx) \leq 1.$$

Hence we have from (17), if k is large enough,

$$\begin{aligned}&\int_{R^N} \xi^2 u_k^{m(2r-1)+1}(x,t)dx + \int_0^t\int_{R^N} \xi^2|\nabla u_k^{mr}|^2 dx\,ds \\ \leq\ & C_1\int_0^t\int_{R^N} |\nabla\xi|^2 u_k^{mr} dx\,ds + C_1.\end{aligned}$$

Thus we can use an analogous argument to Lemma 3 to obtain (16). □

Proof of Theorem 1 By Lemma 1 – Lemma 4, the solution u_k is uniformly bounded for k on every compact set K of $\bar{S}\setminus(0,0)$. Thus if $m \geq 1$, by [9] there exists a subsequence $\{u_{k_j}\}$ and a function $U \in C(\bar{S}\setminus(0,0)$ such that for every compact set $K \subset \bar{S}\setminus(0,0)$

$$u_{k_j} \to U \quad \text{as } k_j \to \infty \text{ in } C(K). \tag{18}$$

If $m < 1$, by [10] there exists a subsequence $\{u_{k_j}\}$ and a function $U \in C(S)$ such that for every compact set $K \subset S$

$$U_{k_j} \to U \quad \text{as } K_j \to \infty \text{ in } C(K). \tag{19}$$

We now prove $U \in C(\bar{S}\setminus(0,0))$ and $U(x,0) = 0$ if $x \neq 0$. For $x_0 \neq 0, t_0 > 0$ let

$$Q(x_0,t_0) = \{(x,t); | x - x_0 |< \frac{| x_0 |}{2}, 0 < t < t_0\}.$$

Since $\overline{Q(x_0,t_0)}$ is a compact set of $\bar{S}\setminus(0,0)$, by Lemma 4, there extsts a consatnt M independent of k such that

$$\sup_{Q(x_0,t_0)} u_k \leq M.$$

Let $g \in C^2(R^N), g(x) > 0, g(x) = 2\varepsilon$ if $| x - x_0 |< \frac{|x_0|}{4}$ and $g(x) = M$ if $| x - x_0 |\geq \frac{|x_0}{2}$. We consider the Dirichlet problem

$$\frac{\partial \omega}{\partial t} = \Delta\omega^m \quad in Q(x_0, t_0), \tag{20}$$

$$\omega(x,0) = g(x) \quad \text{in } \{x :| x - x_0 |< \frac{| x_0 |}{2}\}, \tag{21}$$

$$\omega(x,t) = M \quad \text{in } \{(x,t) :| x - x_0 |= \frac{| x_0 |}{2}, , 0 < t < t_0\}. \tag{22}$$

(20)–(22) have a unique solution $\omega(x,t) \in C^2(\overline{Q(x_0,t_0)})$. Note that if k is large enough

$$k^{\frac{2}{p-m}}\varphi(kx) \leq 2\varepsilon \text{ in } \{x :| x - x_0 |< \frac{| x_0 |}{2}\}.$$

By the comparasion theorem in [7], we have

$$u_k \leq \omega(x,t) \text{ in } Q(x_0,t_0).$$

Hence let $k \to \infty$, to obtain

$$\limsup_{(x,t)\to(x_0,0)} U(x,t) \leq \lim_{(x,t)\to(x_0,0)} \omega(x,t) = 2\varepsilon.$$

This implies

$$\lim_{(x,t)\to(x_0,0)} U(x,t) = 0.$$

Thus $U \in C(\tilde{S} \setminus (0,0)), U(x,0) = 0$ if $x \neq 0$.Clearly U satisfies (7) in the sense of distributions. We now prove

$$\lim_{t \to 0} \int_{B_R} U(x,t)dx = +\infty \quad \text{for every} \quad R > 0.$$

By H1, we can assume that

$$\varphi(x) \geq \alpha_0 > 0 \quad \text{if} \quad |x| < C_0.$$

Let

$$\psi(x,a,k) = \begin{cases} k^N(a + bk^2|x|^2)^{\frac{1}{m-1}} & \text{if } |x| \leq \frac{C_0}{k}, \ m > (1 - \frac{2}{N}), m \neq 1, \\ ak^N \exp\{-\frac{1}{4}k^2|x|^2\} & \text{if } |x| \leq \frac{C_0}{k} \ \ m = 1; \end{cases}$$

$$\psi(x,\alpha,k) = 0, \quad \text{if} \ |x| > \frac{C_0}{k},$$

where $b = \frac{1-m}{2m^2N - 2mN + 4m}, a > bC_0^2, a > 0$. From $\frac{2}{p-m} > N$, we have

$$k^{\frac{2}{p-m}} \varphi(kx) \geq \psi(x,\alpha,k) \quad \text{if } k \text{ large enough.}$$

Let ω_{ak} be the solution of (8) with initial

$$\omega_{ak}(x,0) = \psi(x,a,k). \tag{23}$$

By the Comparision Principle [7], if k is large enough

$$u_k \geq \omega_{ak}. \tag{24}$$

Note that

$$\psi(x,a,k) \leq E_a(x, k^{-N\sigma}), \quad \delta = m - 1 + \frac{2}{N},$$

and

$$\int_{R^N} \psi(x,a,k)dx = \int_{|x|<\frac{C_0}{k}} k^N(a + bk^2|x|^2)^{\frac{1}{m-1}} dx$$

$$= \int_{|y|<C_0} (a + b|y|^2)^{\frac{1}{m-1}} dy, \quad \text{if } m > (1 - \frac{2}{N})^+, \ m \neq 1,$$

$$\int_{R^N} \psi(x,a,k)dx = \int_{|x|<\frac{C_0}{k}} ak^N \exp\{-\frac{1}{4}k^2|x|^2\}dx$$

$$= \int_{|y|<C_0} a\exp\{-\frac{|y|^2}{4}\}dy \quad \text{if} \quad m = 1,$$

where

$$E_a(x,t) = \begin{cases} t^{-\frac{1}{\delta}}\left[a + \frac{(1-m)|x|^2}{2mN\delta t^{\frac{2}{N\delta}}}\right]^{\frac{1}{m-1}} & \text{if } m > (1 - \frac{2}{N})^+, m \neq 1, \\ at^{-\frac{N}{2}} \exp\{-\frac{|x|^2}{4t}\} & \text{if } m = 1, \end{cases}$$

is the Barenblatt–Pattle solution of $u_t = \Delta u^m$. By the Comparision Principle

$$\omega_{ak}(x,t) \le E_a(x, k^{-N\delta} + t). \tag{25}$$

Hence, as has been proved above, there exists a subsequence ω_{ak_j} such that for every compact set $K \subset \bar{S} \setminus (0,0)$

$$\omega_{ak_j} \to \omega_a \ \text{ as } k_j \to \infty.$$

The limit function ω_a is defined and continuous on $\bar{S} \setminus (0,0)$ and by (26), (25)

$$\omega_a \le E_a(x,t). \tag{26}$$

$$\omega_a(x,t) \le U(x,t). \tag{27}$$

Moreover by the definition of solution for $\chi \in C_0^\infty(R^N), \chi \ge 0$

$$\begin{aligned} & \left| \int_{R^N} \omega_{ak_j}(x,t)\chi(x)dx - \int_{R^N} \psi(x,a,k_j)\chi(x)dx \right| \\ \le & \left| \int_0^t \int_{R^N} \omega_{ak_j}^m \Delta\chi dx\, ds \right| + \left| \int_0^t \int_{R^N} k_j^{-\frac{2p}{p-m}} f(k_j^{-\frac{2p}{p-m}} \omega_{ak_j}) \chi(x) dx\, ds \right| \end{aligned} \tag{28}$$

Note that by (25)

$$k^{-\frac{2}{p-m}} \omega_{ak} \le k^{-\frac{2}{p-m}} E_k(x, k^{-N\delta} + t).$$

Thus, if $k \to \infty$,

$$\begin{aligned} & k^{-\frac{2}{p-m}} \omega_k \\ \le & k^{-\frac{2}{p-m}+N} \cdot \left[a + \frac{(1-m)|x|^2}{2mN\delta(t+k^{-N\delta})^{\frac{2}{N}}}\right]^{\frac{1}{m-1}} \to 0 \ \text{ if } m \ne 1, \end{aligned}$$

$$\begin{aligned} & k^{-\frac{2}{p-m}} \omega_{ak} \\ \le & a k^{-\frac{2}{p-m}+N} \cdot \exp\left\{-\frac{|x|^2}{4(t+k^{-2})}\right\} \to 0, \ \text{ if } m = 1, \end{aligned}$$

By H2, if k is large enough, we have

$$k^{\frac{2p}{p-m}} f(k^{-\frac{2p}{p-m}} \omega_{ak}) \le (\delta+1) E_a^p(x, t + k^{-N\delta}).$$

Hence if k_j is large enough, we get from (25) and (28)

$$\begin{aligned} & \left| \int_{R^N} \omega_{ak_j}(x,t)\chi(x)dx - \int_{R^N} \psi(x,a,k_j)\chi(x)dx \right| \\ \le & C\left\{\int_0^t \int_{R^N} [E_a^m(x, k_j^{-N\delta} + s) + (\delta+1) E_\delta^p(x, t + k_j^{-N\delta})\right\} \\ \le & C \int_0^{t+k_j^{-N\delta}} \int_{R^N} [E_a^m(x,s) + (\delta+1) E_a^p(x,s)] dx\, ds. \end{aligned}$$

Letting $k_j \to \infty$, if $m \neq 1$, we obtain

$$\left| \int_{R^N} \omega_a(x,t)\chi(x)dx - \chi(0)\int_{|y|<C_0} (a+b|y|^2)^{\frac{1}{m-1}} dy \right| \leq C\int_0^t \int_{R^N} (E_a^m + E_a^p) dx\, ds.$$

Thus

$$\lim_{t\to 0}\int_{R^N} \omega_a(x,t)\chi(x)dx = \chi(0)\int\int_{|y|<C_0} (a+b|y|^2)^{\frac{1}{m-1}} dy,$$

and by (27)

$$\lim_{t\to 0}\int_{|x|<R} U(x,t)dx \geq \lim_{t\to 0}\int_{|x|<R} \omega_a(x,t)dx \geq \int_{|y|<C_0} (a+b|y|^2)^{\frac{1}{m-1}} dy.$$

Analogously, if $m = 1$ we have

$$\lim_{t\to 0}\int_{|x|<R} U(x,t)dx \geq \lim_{t\to 0}\int_{|x|<R} \omega_a(x,t)dx \geq \int_{|y|<C_0} a\exp\{-\frac{|y|^2}{4}\} dy.$$

Thus if $m \geq 1$, letting $a \to \infty$, we get

$$\lim_{t\to 0}\int_{|x|<R} U(x,t)dx = +\infty.$$

If $m < 1$, we let $a \to 0$ to obtian

$$\lim_{t\to 0}\int_{|x|<R} U(x,t)dx = +\infty.$$

Thus $U(x,t)$ is a very singular solution of (7). By the uniqueness of very singular solution [11], for every compact set $K \subset \bar{S} \setminus (0,0)$,

$$u_k(x,t) \to U(x,t) \quad \text{as} \quad k \to \infty \text{in} \quad C(K).$$

Set $t = 1$ in (18) and (19). Then

$$u_k(x,1) = K^{\frac{2}{p-m}} u(kx, k^\beta) \to U(x,1) \quad \text{as } k \to \infty.$$

uniformly on compact subset of R^N. Thus writing $kx = x', k^\beta = t'$ and dropping the primes again, by $U(xt^{-\frac{1}{\beta}}, 1) = t^{\frac{1}{p-1}} U(x,t)$ we get

$$t^{\frac{1}{p-1}} u(x,t) \to U(xt^{-\frac{1}{\beta}}, 1) = t^{\frac{1}{p-1}} U(x,t) \quad \text{as} \quad t \to \infty$$

uniformly on sets

$$\{x \in R^N : |x| < at^{\frac{1}{\beta}}\} \quad a > 0,$$

and Theorem 1 is proved.

Remark Clearly, the proof of Theorem 1 gives a method to prove the existence of the very singular solution of (7).

References

[1] L.A. Peletier, *The porous medium equation,* Application of Nonilinear Analysisi in the Physical Sciences, Pitman, London, (1981),229– 241.

[2] M. Mertsch, R.Kersner and L.A.Peletier,*Positivity versus localization in degenerate diffusion equations*, Nonlinear Anal. TMA **9** (1985), 987–1008.

[3] A.S. Kalashnikov, *The propagation of distribances of nonlinear heat conduction with absorption*, USSR. Comp. Math. Phys.,**14**(1974), 70–85.

[4] S. Kamin and P.L. Peletieier, *Large time behavior of solution of the porous medium equation with absorption*, Isreal J. of Math., Vol.55, No.2(1986), 129–146.

[5] H. Brezis, L.A. Peletier and D. Terman, *A very singular solution of the heat equation with absorption*, Arch. Rational Mech. Anal., **95**(1986), 185–230.

[6] H. Brezis and M.G. Crandall, *Uniqueness of solutions of the initial value problem for* $u_t - \Delta\phi(u) = 0$, J. Math. Pures et Appl., **58**(1979),153–163.

[7] L.A. Peletier and Zhao Junning, *Source-type solutions of the porous media equation with absorption in the fast diffusion case*, to appear in Nonlinear Analysis.

[8] O.A. Ladyzenskaja, and V. A. Uralceva, *Linear and quasi-linear equations of parabolic type*, Amer. Math. Soc. Transl. Math. Mono., Vol.23.

[9] E. Dibenedeto, *Continuity of weak solutions to a general proous media equation*, Indiana Univ. Math. J., **32**(1983), 83– 118.

[10] P.E. Sacks, *Continuity of solutions of a singular parabolic equation*, Nonlinear Analysis, **7**(1983), 387–409.

[11] S. Kamin and L. Veron, *Exisience and uniqueness of the very singular solution o f the porous media equation with absorption*, to appear.

具吸收项的多孔隙介质方程解的渐近性

杨金顺　　赵俊宁

（东北电力学院数学教研室，吉林市 132021）　（吉林大学数学系，长春市 130023）

本文研究了具吸收项的多孔隙介质方程解的渐近性质，部分地解答了 *P. L. Peletier* 等人关于此问题的一个猜测[4].