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# Some Theorems for the Ranges of Perturbed $m-$accretive Operators*

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**1. Introduction** In recent years, many authors have investigated the ranges of pertubed accretive mappings and obtained some interesting results (e.g., see [2-7]). In [3], Kartsatos proved the following theorem concerning the ranges of compact pertubations of $m-$accretive:

**Theorem A** *Let $X$ be a Banach space, $T: D(T) \subset X \longrightarrow X$ an $m-$accretive operator with $0 \in D(T)$ and $T(0) = 0$, and $C: X \longrightarrow X$ be compact. Suppose that*

*(i)* $\lim_{n\to\infty} \inf\{\sup_{\|x\|\leq n}(1/n) \| Cx \|\} = 0$,

*(ii) there exists a constant $r > 0$ such that*

$$\| C(0) \| < r \leq \lim_{\substack{x \longrightarrow \infty \\ x\in D(T)}} \inf \| Tx + Cx \|,$$

*(iii) there exists $r_1 > 0$ such that for every $x \in D(T)$ with $\| x \| \geq r_1$ there exists $j \in J(x)$ such that $Re\langle Tx + Cx - C(0), j\rangle \geq 0$.*

*Then if $(T+C)(D(T))$ is closed in $X$, $B(0,\mu) \subset R(T+C)$, where $\mu = (r- \| C(0) \|)/2$.*

The purpose of this paer is to obtain some new results for the ranges of pertubed $m-$accretive operators. First, we prove that $\overline{B}(0,r) \subset R(T+C)$ without the assumptions of condition (i) and single-valued for $T$ in Theorem A. Since $\mu \leq r/2$, it is clear that the result is much stronger than Theorem A. Second, we obtain a theorem for zeros of compact pertubations of strongly accretive operators. Finally, we prove that the sum of an $m-$accretive ooperator and a demicontinuous accretive operator is $m-$accretive.

## 2. Preliminaries

The letter $X$ denotes a real or complex Banach space with norm $\| \cdot \|$ and dual $X^*$. The duality mapping $J: X \longrightarrow 2^{X^*}$ is defined by

$$Jx = \{f \in X^* : \langle x, f\rangle = \| x \|^2, \| f \| = \| x \|\},$$

where $\langle x, f\rangle$ denotes $f(x)$. If $x^*$ is uniformly convex, then $J$ is single-valued and uniformly continuous on bounded subset of $X$. The symbols $D(T)$, $R(T)$ denote the domain and

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the range of an operator $T$, respectively. An operator $T : D(T) \subset X \longrightarrow X$ is called demicontinuous if it is continuous on $D(T)$ from the strong topology of $X$ to the weak topology of $X$. $T$ is said to be compact if it is continuous on $D(T)$ and maps bounded subsets of $D(T)$ into relativley compact subset of $X$. $T$ is said to be strongly accretive if there exists $\alpha > 0$ such that for every $x, y \in D(T)$ there exists $j \in J(x-y)$ such that $\mathrm{Re}\langle Tx - Ty, j\rangle \geq \alpha \| x - y \|^2$. $T$ is accretive if $\alpha = 0$ $T$ is $m-$accretive if it is accretive and such that $R(T + \lambda I) = X$ for every $\lambda > 0$, where $I$ denotes the indentity operator.

Let $T$ be $m-$accretive, we define $J_N : X \longrightarrow D(T)$, the resolvent of $T$, by $J_n = (I + \frac{1}{n}T)^{-1}$ and $T_n : X \longrightarrow R(T)$, the Yosida approximation of $T$, by $T_n = n(I - J_n)$. We know that $T_n x \in T(J_n x)$ for every $x \in X$ and $\| T_n x \| \leq | Tx |$ for every $x \in D(T)$. where $| Tx | = \inf\{\| y \| : y \in Tx\}$. We also know that the mappings $J_n$ are nonexpansive and the mappings $T_n$ are $m-$accretive and Lipschitz continuous with Lipschitz constant $2n$.

We denote by " $\rightarrow$ "(" $\rightharpoonup$ ") strong (weak) convergence. We also denote by $B(0, r)$ the open ball with center at zero and radius $r > 0$. The symbols $\overline{D}$,$\partial D$ denote the closure and the boundary of set $D$, respectively. We refer to [2] for the degree theory of compact pertubations of $m-$accretive operators.

## 3.Main Results

**Theorem 1** *Let $X$ be a banach space, $T : D(T) \subset X \longrightarrow 2^X$ an $m-$accretive operator with $0 \in D(T)$ and $0 \in T(0)$, and let $C : X \longrightarrow X$ be compact. Suppose that*

*(i) there exists a constant $r > 0$ such that*

$$\| C(0) \| < r \leq \liminf_{\substack{\|x\| \to \infty \\ x \in D(T)}} | Tx + Cx |,$$

*(ii) there exists $r_1 > 0$ such that for every $x \in D(T)$ with $\| x \| \geq r_1$ there exists $j \in J(x)$ such that*

$$Re\langle y + Cx - C(0), j\rangle \geq 0, \text{ where } y \in Tx.$$

*Then if $(T + C)(D(T)$ is closed in $X$, $\overline{B}(0, r) \subset R(T + C))$.*

**Proof** Let $f \in B(0, r)$. From condition (i), there exists $\varepsilon > 0$ and $R > r_1$ such that for $x \in \partial B(0, R) \cap D(T)$,

$$| Tx + Cx | \geq r - \varepsilon > \max\{\| f \|, \| C(0) \|\}. \tag{1}$$

By Theorem 3.3 in [2], $\deg(\lambda I + T, B(0, s) \cap D(T), 0) = 1$ for every $\lambda > 0, s > 0$.

Let $W(t, x) = Tx + \lambda x + (1-t)(Cx - C(0))$, where $x \in \overline{B}(0, R) \cap D(T)$ and $t \in [0, 1]$.By condition (ii), we know that $0 \notin W(t, \cdot)(\partial B(0, R))$ for $t \in [0, 1]$. By Theorem 3.4 in [2],

$$\deg(\lambda I + T + C - C(0), B(0, R) \cap D(T), 0) = \deg(\lambda I + T, B(0, R) \cap D(T), 0) = 1.$$

Hence

$$\deg(T + C - C(0), B(0, R) \cap D(T), 0) = \lim_{\lambda \to 0} \deg(\lambda I + T + C - C(0), B(0, R) \cap D(T), 0) = 1.$$

Let $H(t,x) = Tx + Cx - tC(0) - (1-t)f$, where $x \in \overline{B}(0,R) \cap D(T), t \in [0,1]$. By inequality (1), $0 \notin \overline{H(t,\cdot)(\partial(B)(0,R) \cap D(T))}$ for $t \in [0,1]$. Using Theorem 4.4 in [2], we obtain

$$\deg(T + C - f, B(0,R) \cap D(T), 0) = \deg(T + C - C(0), B(0,R) \cap D(T), 0) = 1.$$

Consequently, $f \in \overline{(T+C)(B(0,R) \cap D(T))} \subset \overline{R(T+C)}$. Since $R(T+C)$ is closed, $\overline{B}(0,R) \subset \overline{R(T+C)} = R(T+C)$. □

**Theorem 2** *Let $X$ be a Banach space with uniformly convex dual $X^*$, $T : X \longrightarrow X$ a bounded, demicontinuous and strongly accretive operator, $D \subset X$ a bounded open set and let $C : \overline{D} \longrightarrow X$ be a compact opertor. Suppose that for some $z \in D$,*

$$Tx + Cx \neq t(x - z), \text{where } \ x \in \partial D \ \text{ and } \ t < 0. \tag{2}$$

*Then $T + C$ has a zero in $\overline{D}$.*

**Proof** Without loss of generality,, we may assume that $z = 0$ in (2). For each positive integer $m$, let $W(x,t) = \frac{1}{n}x + tTx + tCx(1-t)x$, where $x \in \overline{D}$ and $t \in [0,1]$. From the inequality (2), $0 \notin W(t,\cdot)(\partial D)$ for $t \in [0,1]$. By Theorem 3.4 in [2], we have

$$\deg(\frac{1}{n}I + T + C, D, 0) = \deg((\frac{1}{n} + 1)I, D, 0) = 1.$$

Hence for each $n$, there exists a point $x_n \in D$ such that

$$\frac{1}{n}x_n + Tx_n + Cx_n = 0. \tag{3}$$

Since $C$ is compact, we may assume that the sequence $\{Cx_n\}$ is convergent withut loss of generality. From the strongly accretivity of $T$, there is a constant $\alpha > 0$ such that

$$\begin{aligned} \alpha \| x_n - x_m \|^2 &\leq \mathrm{Re}\langle Tx_n - Tx_m, J(x_n - x_m)\rangle \\ &= -\mathrm{Re}\langle \frac{1}{n}x_n - \frac{1}{m}x_m + Cx_n - Cx_m, J(x_n - x_m)\rangle \\ &\to 0 (m, n \to \infty). \end{aligned}$$

Consequently,there exists a point $x_0 \in \overline{D}$ such that $x_n \longrightarrow x_0 (n \longrightarrow \infty)$. By the demicontinuity of $T$ and the equality (3), we have $Tx_0 + Cx_0 = 0$. □

**Remark 1** *In the case where $C = 0$, the analogous result for $T$ can be found in [7].* From Theorem 2, we have the following corollary.

**Corollary 1** *Let $X$ be a Banach space with uniformly convex dual $X^*$, $T : X \longrightarrow X$ a bounded, demicontinuous and strongly accretive operator, $D \subset X$ a bounded open set with $0 \in D$, and let $C : \overline{D} \longrightarrow X$ be a compact operator. Suppose that there exists constant $r > 0$ such that for $x \in \partial D$,*

$$Re\langle Tx + Cx, Jx\rangle \geq r \| x \| .$$

*Then* $\overline{B}(0,R) \subset (T+C)(\overline{D})$.

**Lemma 1** *Let $X$ be a Banach space with uniformly convex dual $X^*$, and let $T : D(T) \subset X \to 2^X$ be $m-$accretive. If $x_n \in X, n = 1,2,\ldots, x_n \to u \in X$, and $\| T_n x_n \|$ is bounded, then $u \in D(T)$ and there exists a subsequence $\{T_{n_k} x_{n_k}\}$ of $\{T_n x_n\}$ such that $T_{n_k} x_{n_k} \rightharpoonup f \in Tu$.*

The proof is similar to that of Lemma 2.5 in [5], we omit it here.

**Theorem 3** *Let $X$ be a Banach space with uniformly convex dual $X^*$, $T : D(T) \subset X \to 2^X$ an $m-$accretive operator, and let $T_0 : X \to X$ be a bounded and demicontinuous accretive operator. Then $T + T_0$ is $m-$accretive.*

**Proof** It is sufficient to show that for each $\lambda > 0$, $0 \in R(T + T_0 + \lambda I)$ in the case where $X$ is a real Banach space.

Without loss of generality, we may assume that $0 \in D(T)$. For each positive integer $n$ and $t > 0$,

$$\begin{aligned}\langle T_n x + Tx + x + tx, Jx\rangle &\geq \lambda \| x \|^2 + \langle T_n(0), Jx\rangle + \langle T_0(0), Jx\rangle \\ &\geq \lambda \| x \|^2 - | T(0) | \| x \| - \| T_0(0) \| \| x \| .\end{aligned}$$

Hence there exists a constant $R > 0$ such that for $x \in \partial B(0,R)$ and $t > 0$, $\| T_n x + T_0 x + \lambda x + tx \| \geq 0 (n = 1,2,\cdots)$. By the strongly accretivity of $T_n + T_0 + \lambda I$ and Theorm 1 in [7], we have that there exists a point $x_n \in \overline{B}(0,R)$ for each $n$ such that

$$T_n x_n + T_0 x_n + \lambda x_n = 0. \tag{4}$$

Since

$$\begin{aligned}\lambda \| x_n - x_m \|^2 &= \lambda \langle x_n - x_m, J(x_n - x_m)\rangle \\ &= -\langle T_n x_n - T_m x_m, J(x_n - x_m)\rangle - \langle T_0 x_n - T_0 x_m, J(x_n - x_m)\rangle \\ &\leq -\langle T_n x_n - T_m x_m, J(J(_n x_n - J_m x_m)\rangle + \\ &\quad \langle T_n x_n - T_m x_m, J(J_n x_n - J_m x_m)\rangle - \langle T_n x_n - T_m x_m, J(x_n - x_m)\rangle \\ &\leq \| T_n x_n - T_m x_m \| \| J(J_n x_n - J_m x_m) - J(x_n - x_m) \| .\end{aligned}$$

By the equation (4) and boundedness of $T_0$, the sequence $\| T_n x_n \|$ is bounded. Hence $\| J_n x_n - x_n \| \leq \frac{1}{n} \| T_n x_n \| \longrightarrow 0 (n \to \infty)$, which implies from the uniformly continuity of $J$ on bounded sets that $\{x_n\}$ is a Cauchy sequence. Consequently, there is a point $x_0 \in \overline{B}(0,R)$ such that $x_n \to x_0$ $(n \to \infty)$. By Lemma 1, we know that $x_0 \in D(T)$ and there exists s subseuence $\{x_{n_k}\}$ of $\{x_n\}$ such that $T_{n_k} x_{n_k} \rightharpoonup f \in Tx_0$. By the demicontinuity of $T_0$ and the equation (4), $f + T_0 x_0 + \lambda x_0 = 0$. i.e., $0 \in R(T + T_0 + \lambda I)$. □

**Remark 2** *In the case where $T_0$ is locally Lipschitz continuous or $T$ is single-valued and $\overline{D(T)} = X$, Theorem (3.5) and Theorem (10.4) in [1] guarantee the $m-$accretivity of $T + T_0$. However, the above conditions are quite restrictive, we replace the conditons by the boundedness of $T_0$ and show that $T + T_0$ is $m-$accretive.*

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# 关于 $m$—增生算子值域的某些定理


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本文给出几个关于扰动 $m$—增生算子值域的定理,它们改进了文献[1,3,7]中相应的结果.