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Boundedness of Littlewood–Paley Operators and Marcinkiewicz Integral on $\mathcal{E}^{\alpha,p}$ *

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Abstract. In this paper, we shall show that if $f \in \mathcal{E}^{\alpha,p}$, then either $g(f)(x)$ $(s(f)(x), g_\lambda^*(f)(x), \mu(f)(x)) < \infty$ almost everywhere, or $g(f)(x)$ $(s(f)(x), g_\lambda^*(f)(x), \mu(f)(x)) = \infty$ almost everywhere. Furthermore, if $g(f)(x)$ $(s(f)(x), g_\lambda^*(f)(x), \mu(f)(x)) < \infty$ almost everywhere, then $g(f)$ $(s(f), g_\lambda^*(f), \mu(f)) \in \mathcal{E}^{\alpha,p}$ and there is a constant C independent of f and x, so that

$$\| g(f) \|_{\alpha,p} \ (\| s(f) \|_{\alpha,p}, \ \| g_\lambda^*(f) \|_{\alpha,p}, \ \| \mu(f) \|_{\alpha,p}) \leq C \| f \|_{\alpha,p}.$$

§1. Introduction

As we know, Littlewood–Paley operators, Marcinkiewicz integral [5,6], play important role in the classical theory of harmonic analysis.

For $x \in R^n, y > 0$, the Littlewood–Paley functions, $g(f), g_\lambda^*(f)$ and $s(f)$ are defined by

$$g(f)(x) = \{\int_0^\infty y| \operatorname{grad}(f(x,y)) |^2 dy\}^{\frac{1}{2}},$$

$$s(f)(x) = \{\int\int_{\Gamma(x)} y^{1-n}| \operatorname{grad}(f(x,y)) |^2 dz\, dy\}^{\frac{1}{2}},$$

$$g_\lambda^*(f)(x) = \{\int\int_{R_+^{n+1}} (y/(y+ | z-x |))^{\lambda n} y^{1-n}| \operatorname{grad}(f(z,y)) |^2 dz\, dy\}^{\frac{1}{2}} \quad (\lambda > 1).$$

Where $f(x,y)$ is the Poisson integral of f and $\Gamma(x) = \{(z,y) \in R_+^{n+1} :| z-x |< y\}$. The Marcinkiewicz integral is defined by

$$\mu(f)(x) = \{\int_0^\infty [| F(x,t)) |^2/t^3] dt\}^{\frac{1}{2}},$$

where

$$F(x,t) = \int_{|y|\leq t} \left[\Omega(y)/| y |^{n-1}\right] f(x-y) dy,$$

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and Ω satisfies the following two conditions:

(1). Ω is continuous on s^n (s^n is the unit sphere in R^n), and satisfies a Lipschitz condition of order m, $(0 < m < 1)$, $\Omega(tx) = \Omega(x)$ $(x \neq 0, t > 0)$.

(2). $\int_{s^n} \Omega(x')ds^n = 0$.

Let us recall that a locally integrable function $f(x), x \in R^n$, belongs to $\mathcal{E}^{\alpha,p}$ $(1 \leq p < \infty, -n/p \leq \alpha < 1)$ (see [1] & [7]), if there is a constant C, such that for every cube Q,

$$\int_Q | f(x) - f_Q |^p dx \leq C | Q |^{1+\alpha p/n}, \tag{1}$$

where $f_Q = (1/ | Q |) \int_Q f(x)dx$.

The smallest constant $C^{1/p}$ for which C satisfies (1) is called the $\mathcal{E}^{\alpha,p}$ norm of f and is denoted by $\| f \|_{\alpha,p}$, i.e.,

$$\| f \|_{\alpha,p} = \inf\{C^{1/p} : \ C \text{ satisfies } (1)\}. \tag{2}$$

In this paper, we prove the following theorems.

Theorem 1 *Let $f \in \mathcal{E}^{\alpha,p}$ $(\alpha \neq 0, 1 < p < \infty, -n/p \leq \alpha < 1/2)$, then either $g(f)(x) = \infty$ a.e., or $g(f)(x) < \infty$ a.e. $x \in R^n$ and there is a constant C independent of f, such that*

$$\| g(f) \|_{\alpha,p} \leq C \| f \|_{\alpha,p}.$$

Theorem 2 *Let $f \in \mathcal{E}^{\alpha,p}$ $(\alpha \neq 0, 1 < p < \infty, -n/p \leq \alpha < 1/2)$, then either $s(f)(x) = \infty$ a.e., or $s(f)(x) < \infty$ a.e. $x \in R^n$ and there is a constant C independent of f, such that*

$$\| s(f) \|_{\alpha,p} \leq C \| f \|_{\alpha,p}.$$

Theorem 3 *Let $f \in \mathcal{E}^{\alpha,p}$ $(\alpha \neq 0, 1 < p < \infty, -n/p \leq \alpha < 1/2)$, $\lambda > \max(1, 2/p)$, then either $g(f)(x) = \infty$ a.e., or $g(f)(x) < \infty$ a.e. $x \in R^n$ and*

$$\| g^*_\lambda(f) \|_{\alpha,p} \leq C \| f \|_{\alpha,p},$$

where C is a constant independent of f.

Theorem 4 *Let $f \in \mathcal{E}^{\alpha,p}$ $(\alpha \neq 0, 1 < p < \infty)$ and $-n/p \leq \alpha < \min(1/2, m)$, then either $\mu(f)(x) = \infty$ a.e., or $\mu(f)(x) < \infty$ a.e. $x \in R^n$ and*

$$\| \mu(f) \|_{\alpha,p} \leq C \| f \|_{\alpha,p},$$

where C is a constant indenpent of f.

For the case of $\alpha = 0$, $\mathcal{E}^{\alpha,p}$ =BMO, D.S. Kurtz [2], Wang [3] and Han [4] have obtained the similar results.

We also use χ_E to denote the characteristic function of the measurable set E, and for a cube Q, dQ stands for another cube concetric with Q and having edge length d times as long. Also, C denotes some constants which are independent of f and may change from line to line.

§2. The Key Lemma

Observing the proof of the main theorems in [2], [3] & [4], the authors all used an important Lemma [2, Lemma 1.1, p.659] and its special cases.

Here, in order to prove the theorems, we prove the following

Lemma *Let $f \in \mathcal{E}^{\alpha,p}$ $(\alpha \neq 0, 1 < p < \infty, -n/p \leq \alpha < 1)$ and Q is a cube in R^n centered at $\bar{x}$ and having edge length r, for a given real positive number d, suppose that $\alpha < d$, then there is a constant C depending only on n, p, α and d so that for any arbitrary $y > 0$,*

$$\int_{R^n} \frac{|f(x) - f_Q|}{y^{n+d} + |x - \bar{x}|^{n+d}} dx \leq C y^{-d}(y^\alpha + r^\alpha) \| f \|_{\alpha,p}. \tag{3}$$

In particular,

$$\int_{R^n} \frac{|f(x) - f_Q|}{r^{n+d} + |x - \bar{x}|^{n+d}} dx \leq C r^{\alpha-d} \| f \|_{\alpha,p}, \tag{4}$$

$$\int_{R^n} \frac{|f(x) - f_{Q_0}|}{1 + |x|^{n+d}} dx \leq C \| f \|_{\alpha,p}, \tag{5}$$

where Q_0 is the unit cube.

Proof Arguing as in [1], we can easily obtain the special cases of the Lemma (4) and (5).

Now, our main task is to prove (3).

At first, we claim that if R is a cube having the same center $\bar{x}$ and edge length y, then

$$|f_R - f_Q| \leq C(y^\alpha + r^\alpha) \| f \|_{\alpha,p}. \tag{6}$$

In fact, if $y \geq r$, we can choose nonnegative integer k such that $2^k r \leq 2^{k+1} r$, write $Q(j) = 2^j Q$, $Q(0) = Q$, $(j = 1, 2, \cdots)$, then

$$\begin{aligned} & |f_R - f_Q| \leq (1/|R|) \int_R |f(x) - f_Q| dx \\ \leq \ & (1/|R|) \int_{Q(k+1)} |f(x) - f_Q| dx \\ \leq \ & (1/|R|) \left[\int_{Q(k+1)} |f(x) - f_{Q(k+1)}| dx + |Q(k+1)| \, |f_{Q(k+1)} - f_Q \right] | \\ \leq \ & (1/|R|) \int_{Q(k+1)} |f(x) - f_{Q(k+1)}| dx + |Q(k+1)| \sum_{j=1}^{k+1} |f_{Q(j-1)} - f_{Q(j)}| \end{aligned} \tag{7}$$

but

$$
\begin{aligned}
& \int_{Q(k+1)} | f(x) - f_{Q(k+1)} | \, dx \\
\leq \;& | Q(k+1) |^{1-1/p} \left(\int_{Q(k+1)} | f(x) - f_{Q(k+1)} |^p dx \right)^{1/p} \\
\leq \;& | Q(k+1) |^{1-1/p} | Q(k+1) |^{1/p+\alpha/n} \| f \|_{\alpha,p} \\
\leq \;& | Q(k+1) |^{1+\alpha/n} \| f \|_{\alpha,p} \\
\leq \;& C \| f \|_{\alpha,p} (2^k r)^{n+\alpha}
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
& | f_{Q(j)} - f_{Q(j-1)} | \\
\leq \;& \frac{1}{| Q(j-1) |} \int_{Q(j)} | f(x) - f_{Q(j)} | \, dx \\
\leq \;& C \frac{1}{| Q(j) |} \int_{Q(j)} | f(x) - f_{Q(j)} | \, dx \\
\leq \;& C \frac{1}{| Q(j) |} \int_{Q(j)} | f(x) - f_{Q(j)} |^p dx^{1/p} \\
\leq \;& C | Q(j) |^{-1/p} | Q(j) |^{1/p+\alpha/n} \| f \|_{\alpha,p} \\
\leq \;& C | Q(j) |^{\alpha/n} \| f \|_{\alpha,p} \\
\leq \;& C \| f \|_{\alpha,p} (2^j r)^{\alpha}.
\end{aligned} \tag{9}
$$

By (9), we have

$$
\begin{aligned}
& \sum_{j=1}^{k+1} | f_{Q(j)} - f_{Q(j-1)} | \\
\leq \;& C \| f \|_{\alpha,p} \, r^{\alpha} \sum_{j=1}^{k+1} 2^{j\alpha} \\
\leq \;& C \| f \|_{\alpha,p} \, (2^k r)^{\alpha} \\
\leq \;& C y^{\alpha} \| f \|_{\alpha,p} \, ,
\end{aligned} \tag{10}
$$

and

$$
\sum_{j=1}^{k+1} | f_{Q(j)} - f_{Q(j-1)} | \leq C r^{\alpha} \| f \|_{\alpha,p} \, , \text{ for } \alpha < 0. \tag{11}
$$

Combining of (7), (8), (9), (10) and (11) yields that for $\alpha > 0$,

$$
\begin{aligned}
| f_R - f_Q | \;&\leq\; C y^{-n} [(2^k r)^{n+\alpha} + (2^k r)^n (2^k r)^{\alpha}] \| f \|_{\alpha,p} \\
&\leq\; C y^{-n} (2^k r)^{n+\alpha} \| f \|_{\alpha,p} \;\leq C y^{-n} y^{n+\alpha} \| f \|_{\alpha,p} \\
&\leq\; C y^{\alpha} \| f \|_{\alpha,p} \, ,
\end{aligned} \tag{12}
$$

and for $\alpha < 0$,

$$\begin{aligned} | f_R - f_Q | &\leq Cy^{-n}[(2^k r)^{n+\alpha} + (2^k r)^n r^\alpha] \| f \|_{\alpha,p} \\ &\leq Cy^{-n}[y^{n+\alpha} + y^n r^\alpha] \| f \|_{\alpha,p} \\ &\leq C(y^\alpha + r^\alpha) \| f \|_{\alpha,p} . \end{aligned} \tag{13}$$

From (12) and (13), (5) is immediate.

By (6), (4) and noticing that $\int_{R^n} [1/(y^{n+d} + | x - \bar{x} |^{n+d})] dx \leq cy^{-\alpha}$, then

$$\begin{aligned} \int_{R^n} \frac{| f(x) - f_Q |}{y^{n+d} + | x - \bar{x} |^{n+d}} dx &\leq \int_{R^n} \frac{| f(x) - f_R |}{y^{n+d} + | x - \bar{x} |^{n+d}} dx + \\ &\quad | f_R - f_Q | \int_{R^n} \frac{1}{y^{n+d} + | x - \bar{x}^{n+d}} dx \\ &\leq Cy^{\alpha - d} \| f \|_{\alpha,p} + Cy^{-d}(y^\alpha + r^\alpha) \| f \|_{\alpha,p} \\ &\leq Cy^{-d}(y^\alpha + r^\alpha) \| f \|_{\alpha,p} . \end{aligned}$$

The proof is complete. □

Remark If $d = 1$, then

$$\int_{R^n} [| f(x) - f_{Q_\circ} | / (1 + | x |^{n+1})] dx \leq C \| f \|_{\alpha,p} ,$$

it follows that $\int_{R^n} [| f(x) | / (1 + | x |^{n+1})] dx < \infty$, which is equivalent to the finiteness of the Possion integral $f(x, y)$, $y > 0$, of f.

§3. The Proof of the Theorems

Just as stated in the beginning of Section 2, the proof of the main theorems in [2], [3] & [4] depend closely on the Lemma in [2, p659] and its special forms.

In the proof of our theorems, we repeat the argument of the proof of the corresponding theorems in [2], [3] & [4], and use the Lemma in Section 2 instead of the Lemma in [2, Lemma 1.1, p659]. We only give the proof of Theorems 2 and 3, but not state all the details.For the proof of other theorems in this paper, we can do in a similar way.

Let Tf be an $s-$function $s(f)$ or g_λ^*-function $g_\lambda^*(f)$ $(\lambda > \max(1, 2/p))$, suppose that $| \{Tf \neq \infty\} | > 0$, let $\underline{x}$ be a density point of $E = \{x : Tf(x) < \infty\}$ and Q be any cube centered at $\underline{x}$, write f as

$$f(x) = f_Q + [f(x) - f_Q] x_Q + [f(x) - f_Q] \chi_{cQ} = f_Q(x) + g_Q(x) + h_Q(x).$$

Clearly, $Tf_Q = 0$. Since $f \in \mathcal{E}^{\alpha,p}$ $(1 < p < \infty, -n/p \leq \alpha < 1)$,

$$\begin{aligned} \| g_Q \|_p &= (\int_Q | f(t) - f_Q |^p dt)^{\frac{1}{p}} \\ &\leq C | Q |^{\frac{1}{p} + \frac{\alpha}{n}} \| f \|_{\alpha,p} , \end{aligned} \tag{14}$$

where $Q \in L^p$.

Just like the proof in [2], the main work we need to do is to prove that for sufficiently small d depending only on n, there is a constant C depending only on n, α and $p(\lambda)$ so that for all $x \in dQ$

(i) $Th_Q(x') < \infty \Longrightarrow Th_Q(x) < \infty$,

(ii) $| Th_Q(x) - Th_Q(x') | \leq Cr^{\alpha} \| f \|_{\alpha,p}$, where r is the edge lendgth of Q.

Assume that (i) and (ii) are proved, arguing as in [2], we see that Tf is finite almost everywhere,and we need only to show that

$$\| Tf \|_{\alpha,p} \leq C \| f \|_{\alpha,p} .$$

Let Q' be any cube in R^n and $Q = (1/d)Q'$ $(Q' = dQ)$, choose a point $x' \in dQ$ so that $Th_Q(x') < \infty$. Then, by (14) and (ii),

$$\begin{aligned}
&\left(\int_{Q'} | Tf(x) - Th_Q(x') |^p dx\right)^{1/p} \\
= &\left(\int_{Q'} | T(g_Q + h_Q)(x) - Th_Q(x) + Th_Q(x) - Th_Q(x') |^p dx\right)^{1/p} \\
\leq &\left(\int_{Q'} | Tg_Q(x) |^p dx\right)^{\frac{1}{p}} + \int_{Q'} | Th_Q(x) - Th_Q(x') |^p dx)^{1/p} \\
\leq &\ C \| g_Q \|_p + C | Q' |^{1/p} r^{\alpha} \| f \|_{\alpha,p} \\
\leq &\ C | Q |^{1/p+\alpha/n} \| f \|_{\alpha,p} + C | Q |^{1/p} | Q |^{\alpha/n} \| f \|_{\alpha,p} \\
\leq &\ C | Q |^{1/p+\alpha/n} \| f \|_{\alpha,p}.
\end{aligned}$$

Thus

$$\int_{Q'} | Tf(x) - Th_Q(x') |^p dx \leq C | Q |^{1+\alpha p/n} \| f \|^p_{\alpha,p},$$

this implies that $\| Tf \|_{\alpha,p} \leq C \| f \|_{\alpha,p}$, the proof is complete.

Theorems 2 and 3 are proved modulo the results of the following Claims 1 and 2.

Claim 1 Suppose that $f \in \mathcal{E}^{\alpha,p}$ $(\alpha \neq 0, 1 < p < \infty, -n/p \leq \alpha < 1/2)$. Let Q be a cube with center $\underline{x}$ and edge length r. Set $d = 1/(8\sqrt{n})$. If there is $x' \in dQ$ so that $s(h_q)(x') < \infty$. Then there is a constant C depending only on n, α and p, such that $s(h_Q)(x) < \infty$ and

$$| s(h_Q)(x) - s(h_Q)(x') | \leq Cr^{\alpha} \| f \|_{\alpha,p} \text{ ,for all } x \in dQ.$$

Claim 2 Suppose that $f \in \mathcal{E}^{\alpha,p}$,$(\alpha \neq 0, 1 < p < \infty, -n/p \leq \alpha < 1/2)$ and $\lambda > \max(1, 2/p)$. Q and d stated as in Claim 1. If there is $x' \in dQ$ such that $g^*_\lambda(h_Q)(x') < \infty$. Then there is a constant C depending only on n, α, λ and p, such that $g^*_\lambda(h_Q)(x) < \infty$ and

$$| g^*_\lambda(h_Q)(x) - g^*_\lambda(h_Q)(x') \leq Cr^{\alpha} \| f \|_{\alpha,p} \text{ ,for all } x \in dQ.$$

3.1. Proof of the Claims

Arguing as in [2, Lemma 2.1], we have

$$s(h_Q)(x) \le s^- + s^+, x \in dQ,$$

where

$$s^- = \left(\iint_{\Gamma(x)^-} y^{1-n} |\operatorname{grad}(h_Q(z,y))|^2 dz dy\right)^{\frac{1}{2}},$$

$$\Gamma(x)^- = \{(z,y) \in \Gamma(x) : y \le dr\}$$

and

$$s^+ = \left(\iint_{\Gamma(x)^+} y^{1-n} |\operatorname{grad}(h_Q(z,y))|^2 dz dy\right)^{\frac{1}{2}},$$

$$\Gamma(x)^+ = \{(z,y) \in \Gamma(x) : y > dr\}.$$

Estimate s^- and s^+ as in [2] and use the Lemma in Section 2, we have

$$\begin{aligned} s^- &\le C\left(\iint_{\Gamma(x)^-} y^{1-n}\left(\int_{cQ} \frac{|f(x) - f_Q|}{|t - x|^{n+1} + r^{n+1}} dt\right)^2 dz\, dy\right)^{\frac{1}{2}} \\ &\le C\left(\int_0^{dr}\int_{\{|z-x|<y\}} y^{1-n} r^{-2} r^{2\alpha} \| f \|_{\alpha,p}^2 dz\, dy\right)^{\frac{1}{2}} \\ &\le C r^{\alpha} \| f \|_{\alpha,p} \end{aligned}$$

and

$$\begin{aligned} s^+ &\le s(h_Q)(x') + C\left(\iint_{\Gamma(x)^+ \setminus \Gamma(x')} y^{1-n}\left(\int_{cQ} \frac{|f(t) - f_Q|}{|t - x|^{n+1} + y^{n+1}} dt\right)^2 dz\, dy\right)^{\frac{1}{2}} \\ &\le s(h_Q)(x') + C\left(\iint_{\Gamma(x)^+ \setminus \Gamma(x')} y^{1-n} y^{-2} (y^{\alpha} + r^{\alpha})^2 \| f \|_{\alpha,p}^2 dz\, dy\right)^{\frac{1}{2}} \\ &\le s(h_Q)(x') + C \| f \|_{\alpha,p} \left(\int_{dr}^{\infty} r y^{-2} (y^{2\alpha} + r^{2\alpha}) dy\right)^{\frac{1}{2}} \\ &\le s(h_Q)(x') + C r^{\alpha} \| f \|_{\alpha,p} . \end{aligned}$$

Therefore,

$$s(h_Q)(x) \le s(h_Q)(x') + C r^{\alpha} \| f \|_{\alpha,p} ,$$

thus

$$s(h_Q)(x) < \infty \quad \text{for all} \quad x \in dQ.$$

Reversing the roles of x and x', we get

$$|s(h_Q)(x) - s(h_Q)(x')| \le C r^{\alpha} \| f \|_{\alpha,p} \quad \text{for all } x \in dQ.$$

Claim 1 is true.

3.2 Proof of Claim 2

Arguing as in [2, Lemma 2.2], we have $g_\lambda^*(h_Q)(x)$ $(x \in dQ)$ is bounded by the sum of

$$G^{-}= \Big(\int\int_{J(0)} \big(\frac{y}{y+\mid x-z\mid}\big)^{\lambda n} y^{1-n} \mid \operatorname{grad}(h_Q(z,y)) \mid^2 dz\, dy\Big)^{\frac{1}{2}}$$

and

$$G^{+} == \Big(\int\int_{R_+^{n+1}\setminus J(0)} \big(\frac{y}{y+\mid x-z\mid}\big)^{\lambda n} y^{1-n} \mid \operatorname{grad}(h_Q(z,y)) \mid^2 dz\, dy\Big)^{\frac{1}{2}},$$

where $J(k) = \{(z,y) \in R_+^{n+1} : \mid z - \underline{x} \mid < 2^{k-2}r \text{ and } 0 < y < 2^{k-2}r\}$ for $k \geq 0$.

Repeating the argument of the estimates for G^- and G^+ as in [2], we have

$$\begin{aligned} G^{-} &\leq C\Big(\int\int_{J(0)} \big(\frac{y}{y+\mid x-z\mid}\big)^{\lambda n} y^{1-n} \Big(\int_{cQ} \frac{\mid f(t) - f_Q \mid}{\mid t - \underline{x} \mid^{n+1} + r^{n+1}} dt\Big)^2 dz\, dy\Big)^{\frac{1}{2}} \\ &\leq C\Big(\int_0^r \int_{\{|z-x|<r\}} \big(\frac{y}{y+\mid x-z\mid}\big)^{\lambda n} y^{1-n} r^{-2} r^{2\alpha} \| f \|_{\alpha,p}^2 dz\, dy\Big)^{\frac{1}{2}} \\ &\leq C r^{\alpha} \| f \|_{\alpha,p}, \end{aligned}$$

$$G^{+} \leq g_\lambda^*(h_Q)(x') + \tau,$$

while

$$\tau \leq C\Big(\sum_{k=1}^{\infty} 2^{-k}(2^k r)^{-\lambda n} \int\int_{J(k)\setminus J(k-1)} y^{\lambda n+1-n} \mid \operatorname{grad}(h_Q(z,y)) \mid^2 dz\, dy\Big)^{\frac{1}{2}}.$$

Without loss of generality we may also assume $1 < \lambda < 2$. In this case,

$$\tau \leq C\Big(\sum_{k=1}^{\infty} 2^{-k}(2^k r)^{-\lambda n}(A_k + B_k)\Big)^{\frac{1}{2}},$$

where

$$\begin{aligned} A_k &= \int\int_{J(k)} y^{\lambda n+1-n} \Big(\int_{cQ(k+1)} \frac{\mid f(t) - f_Q \mid}{\mid t - z \mid^{n+1} + y^{n+1}} dt\Big)^2 dz\, dy, \\ B_k &= \int\int_{J(k)} y^{\lambda n+1-n} \Big(\int_{Q(k+1)\setminus Q} \frac{\mid f(t) - f_Q \mid}{\mid t - z \mid^{n+1} + y^{n+1}} dt\Big)^2 dz\, dy. \end{aligned}$$

By the same reason as in [2],

$$\begin{aligned} A_k &\leq C \int\int_{J(k)} y^{\lambda n+1-n} \Big(\int_{cQ(k+1)} \frac{\mid f(t) - f_Q \mid}{\mid t - x \mid^{n+1} + (2^k r)^{n+1}} dt\Big)^2 dx\, dy \\ &\leq C \int_0^{2^k r} \int_{\{|z| \leq 2^k r\}} y^{\lambda n+1-n} \big((2^k r)^{-1}(r^{\alpha} + (2^k r)^{\alpha}) \| f \|_{\alpha,p}\big)^2 dz\, dy \\ &\leq C(2^k r)^{\lambda n}(1 + 2^{2k\alpha}) r^{2\alpha} \| f \|_{\alpha,p}, \end{aligned}$$

and

$$B_k \leq C\Big(\int_{Q(k+1)\setminus Q} \mid f(t) - f_Q \mid^{2/\lambda} dt\Big)^{\lambda}.$$

Let $q = 2/\lambda \geq 1$, then $1 \leq q \leq p$,

$$\begin{aligned}
(\int_{Q(k+1)} | f(t) - \tau_Q |^q dt)^{1/q} &\leq (\int_{Q(k+1)} | f(t) - f_{Q(k+1)} |^q dt)^{1/q} \\
&+ | Q(k+1) |^{1/q} \cdot | f_{Q(k+1)} - f_Q | \\
&\leq | Q(k+1) |^{1/q} \cdot | f_{Q(k+1)} - f_Q | \\
&+ | Q(k+1) |^{1/q - 1/p} (\int_{Q(k+1)} | f(t) - f_{Q(k+1)} |^p dt)^{1/p} \\
&\leq C(2^k r)^{n(1/q-1/p)} (2^k r)^{\alpha + n/p} \| f \|_{\alpha,p}
\end{aligned}$$

$+C(2^k r)^{n/q}((2^k r)^\alpha + r^\alpha) \| f \|_{\alpha,p}$

$$\leq C r^\alpha \| f \|_{\alpha,p} \cdot (2^k r)^{n/q}((2^k)^\alpha + 1),$$

this implies

$$\begin{aligned}
B_k &= C r^{2\alpha} \| f \|_{\alpha,p}^2 (2^k r)^{2n/q} (2^{k\alpha} + 1)^2 \\
&\leq C r^{2\alpha} \| f \|_{\alpha,p}^2 (2^k r)^{n\lambda} (1 + 2^{2k\alpha}).
\end{aligned}$$

Thus,

$$A_k + B_k \leq C r^{2\alpha} \| f \|_{\alpha,p}^2 (2^k r)^{n\lambda} (1 + 2^{2k\alpha}).$$

Since $\alpha < 1/2$,

$$\begin{aligned}
\tau &\leq C(\sum_{k=1}^{\infty} 2^{-k} (1 + 2^{2k\alpha}) r^{2\alpha} \| f \|_{\alpha,p}^2)^{\frac{1}{2}} \\
&\leq C r^\alpha \| f \|_{\alpha,p} (\sum_{k=1}^{\infty} 2^{-k} + \sum_{k=1}^{\infty} 2^{-k(1-2\alpha)})^{\frac{1}{2}} \\
&\leq C r^\alpha \| f \|_{\alpha,p} .
\end{aligned}$$

Therefore,

$$g_\lambda^*(h_Q)(x) \leq g_\lambda^*(h_Q)(x') + C r^\alpha \| f \|_{\alpha,p} ,$$

and $g_\lambda^*(h_Q)(x) < \infty$ for all $x \in dQ$.

Reversing the roles of x and x', we can obtain

$$| g_\lambda^*(h_Q)(x) - g_\lambda^*(h_Q)(x') | \leq C r^\alpha \| f \|_{\alpha,p} .$$

This ends the proof of Claim 2. □

Supplementary Remark: This paper is one part of the author's thesis [9] for Master's Degree completed in August, 1988. Recently, [10] gave the similar results about the "Boundedness of the Littlewood–Paley g–function on $\mathrm{Lip}_\alpha(R^n)$ $(0 < \alpha < 1)$ (i.e.,$\mathcal{E}^{\alpha,p}$ (R^n))", but it did not consider the case for other Littlewood–Paley functions presented here.

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Littlewood—Paley 算子及 Marcinkiewicz 积分在 Campanato 空间 $\varepsilon^{\alpha,p}$ 上的有界性

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我们证明了下述结果:若 $f\in\varepsilon^{\alpha,p}$,则适当限制参数值时,有 $g(f)(x)(S(f)(x),g_\lambda^*(f)(x),\mu(f)(x))<\infty$ a. e. ,或者 $g(f)(x)(S(f)(x),g_\lambda^*(f)(x),\mu(f)(x))<\infty$ a. e. ;并且在前者成立时,有 $g(f)(S(f),g_\lambda^*(f),\mu(f)\in\varepsilon^{\alpha,p}$,以及 $\|g(f)\|_{\alpha,p},(\|S(f)\|_{\alpha,p},\|g_\lambda^*(f)\|_{\alpha,p},\|\mu(f)\|_{\alpha,p})\leqslant C\|f\|_{\alpha,p}$,其中 C 为不依赖于 f 的常数.