Journal of Mathematical Research and Exposition
Vol. 12, No.1, Feb., 1992

Relative Hereditary Rings*

Guo Yu

(Dept. of Math., Changcun Teachers' College,China)

Abstract. In this paper, we give the concept of relative hereditary ring and characterizatin of relative hereditary ring. The relations between relative hereditary rings, semi-hereditary rings and hereditary rings are studied. Finally, we prove that endmorphism rings of left ideal of some relative hereditary rings and endmorphism rings of finitely generated relative projective modules over some relative hereditary rings are semi-hereditary rings.

1. Relative Hereditary Rings

In this paper, R will always denote an associative ring with identity, all modules mean unitary left $R-$modules. We write the module homomorphism f on the right.

Definition 1.1 Let $_RU$ and $_RM$ be two left $R-$modules, then U is projective relative to M (or U is $M-$projective) in case for each epimorphism $g : {}_RM \to {}_RN$ and each homomorphism $h : {}_RU \to {}_RN$ there exists a homomorphism $\bar{h} : {}_RU \to {}_RM$ such that the diagram commutes, i.e., $h = \bar{h}g$. U is injective relative to M (or U is $M-$injective) are

$$\begin{tikzcd} & U \arrow[dl, dashed, "\bar{h}"'] \arrow[d, "h"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

defined by dualism.

By [1], we have

Proposition 1.2 *A module $_RP$ is projective if and only if it is projective relative to every module $_RM$. And a module $_RQ$ is injective if and only if it is injective relation to every module $_RM$.*

Proposition 1.3 *Let M be a left $R-$module and let $(\cup_\alpha)_{\alpha\in A}$ be an indexed set of left $R-$modules. Then*

*Received May 4, 1990.

(1) $\oplus_A \cup_\alpha$ *is* $M-$*projective if and only if each* $\cup_\alpha$ *is* $M-$*projective;*

(2) $\oplus_A \cup_\alpha$ *is projective if and only if each* $\cup_\alpha$ *is projective.*

Proposition 1.4 *Let* U *be a left* $R-$*module.*

(1) If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *is an* $R-$*module exact sequence and* U *is* $M-$*projective, then* U *is* $M'-$*projective and* $M''-$*projective;*

(2) If U *is projective realtive to each of* $M_1, M_2, \cdots, M_n$*, then* U *is* $\oplus_{i=1}^{n} M_i-$*projective.*

Definition 2 Let R be a ring and let M be a left $R-$module. R is said to be a left $M-$hereditary in case every left ideal of R is $M-$ projective.

By Proposition 1.2, 1.4, we have

Proposition 1.5 *A ring* R *is left hereditary if and only if* R *is left* $M-$*hereditary for every left* $R-$*module* M.

Proposition 1.6 *If* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *is an* $R-$*module exact seqnence,* R *is left* $M-$*hereditary, then* R *is left* $M'-$*hereditary and left* $M''-$*hereditary.*

Theorem 1.7 *Let* M *be an injective left* $R-$*module, then the following statements are equivalent*

(1) R *is left* $M-$*hereditary;*

(2) Every factor module of M *is injective;*

(3) Every submodule of an $M-$*projective left* $R-$*module is* $M-$*projective.*

Proof (1) $\Rightarrow$ (2). Suppose that M/K is a factor module of M,$e : {}_RL \to {}_RR$ is a monomorphism. Consider

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "i"] \arrow[d, "h"'] & P \\ 0 \arrow[r] & M' \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & N \arrow[r, "i"] \arrow[d, "h"'] & P \arrow[dl, dashed, "f"'] \arrow[d, dashed, "g"] \\ 0 \arrow[r] & M' \arrow[r, "o"'] & M \end{tikzcd}$$

Here, h is a homomorphism, n_k is natural homomorphism. Since ${}_RL \cong {}_RI \leq R$ and I is $M-$projective by the assumption of (1), L is $M-$projective. Thus, there exists a homomorphism $f : L \to M$ such that $h = fn_k$. But M is injective, so there exists a homomorphism $g : R \to M$ such that $f = eg$.Thus $h = fn_k = egn_k = e(gn_k)$, and hence M/K is $R-$injective. By the injective test Lemma in [1,18.3], M/K is injective.

(2) $\Rightarrow$ (3). Let P be an $M-$projective left $R-$module, $N \leq P, e : M \to M'$ be an epimorphism. Consider

Here, i is canonical injection, h is a homomorphism.Since $M' \cong M/\ker e$ and $M/\ker e$ is injective by the assumption of (2), M' is injective. Thus there exists a homomorphism $f: P \to M'$ such that $h = if$. But P is $M-$projective, so there exists a homomorphism $g: P \to M$ such that $f = ge$, and hence $h = if = ige = (ig)e$. Thus N is $M-$projective.

(3) $\Rightarrow$ (1). Since regular ${}_RR$ is projective,${}_RR$ is M--projective. The result follows from the assumption (3). □

Theorem 1.8 *The following statements about a ring R are equivalent:*

(1) R is left hereditary;

(2) R is left $M-$hereditary for every injective left $R-$module M;

(3) Every factor module of an injective left $R-$module is injective;

(4) R is left $P-$hereditary for every projective left $R-$module P;

(5) R is left $F-$hereditary for every left $R-$module F.

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3). This follows at once from Proposition 1.5 and Theorem 1.7. (3) $\Rightarrow$ (1). We recall that a ring R is left hereditary iff every submodule of a projective left module is projective. Let P be a projective left $R-$module, $N \leq P$, and let E be an injective left $R-$module. Since P is $E-$projective, by the assumption of (3) and Theorem 1.7, N is $E-$projective. But, arbitrary left $R-$module M is a submodule of some injective left $R-$module E, by Proposition 1.4, so N is $M-$projective. And hence N is a projective left $R-$module. (1) $\Rightarrow$ (4). This follows at once from Proposition 1.5. (4) $\Rightarrow$ (5). Since every free left $R-$module is projective, the result follow from assumption (4). (5) $\Rightarrow$ (1). Since every left $R-$module M is an epimorphic image of a free left $R-$module, R is left $M-$hereditary by assumption (5) and Proposition 1.6, 1.5,so R is left hereditary. □

Proposition 1.9 *Let R be a left Artinian ring, $J = J(R)$ be Jacobson radical of R,$e_1, e_2, \cdots, e_n$ be a complete set of pairwise orthogonal primitive idempotents and let M be left $R-$module. Then the following statements are equivalent*

(1) R is left $M-$hereditary;

(2) ${}_RJ$ is left $M-$projective;

(3) Je_i is left $M-$projective, $i = 1, 2, \cdots, n$.

Proof (1) $\Rightarrow$ (2). This is immediate from Definition 2.

(2) $\Rightarrow$ (3). Since $J = Je_1 \oplus Je_2 \oplus \cdots \oplus Je_n$, thus, by Proposition 1.3, Je_i is $M-$projective, $i = 1, 2, \cdots, n$. (3) $\Rightarrow$ (1).

Let I be arbitrary one of left ideal of R, then I is a left Artinian $R-$module, and hence I has a composition series

$$I = I_0 > I_1 > I_2 > \cdots > I_r = 0.$$

Since $R/J = Re_i/Je_i \oplus Re_2/Je_2 \oplus \cdots \oplus Re_n/Je_n$, every Re_i/je_i is simple, and I_k/I_{k+1} is simple left $R-$module for every k, thus I_k/I_{k+1} is isomorphic to a factor of R/J, and so is isomorphic to one of the Re_i/Je_i. Thus

$$0 \to Je_i \to Re_i \to I_k/I_{k+1} \to 0$$

is a short exact sequence. But

$$0 \to I_{k+1} \to I_k \to I_k/I_{k+1} \to 0$$

is also a short exact sequence, by Schanuel's Lemma [2,Lemma 11.28], we have

$$I_{k+1} \oplus Re_i \cong I_k \oplus Je_i.$$

By induction on k and Proposition 1.3 and assumption (3), we know that $I = I_0$ is M–projective, i.e., R is left M–hereditary. □

Corollary 1.10 *Let R be a left Artinian ring, $J = J(R)$ be Jacobson radical of R, $e_1, e_2, \cdots, e_n$ be a complement set of pairwise orthogonal primitive idempotents. Then the following statements are equivalent:*

(1) R is left hereditary;

(2) ${}_RJ$ is projective;

(3) Je_i is projective, $i = 1, 2, \cdots, n$.

Proposition 1.11 *Let R and S be (Morita) equivalent rings via an equivalent functor $F : {}_RM \to {}_SM$ (${}_RM$ and ${}_SM$ are left R–module category and left S–module category resp.) and let M be an injective left R–module. Then R is M–hereditary if and only if S is $F(M)$–hereditary.*

Proof By Proposition 21.4 and Proposition 21.6 in [1], we have the following results

(1) $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact in ${}_RM$ if and only if $0 \to M' \xrightarrow{F(f)} M \xrightarrow{F(g)} M'' \to 0$ is exact in ${}_SM$;

(2) U is M–projective if and only if $F(U)$ is $F(M)$–projective;

(3) U is injective if and only if $F(U)$ is injective.

($\Rightarrow$). Assume that R is M–hereidtary and $G : {}_SM \to {}_RM$ is inverse equivalence if F (i.e.,$GF \cong l_RM, FG \cong l_SM$). Since M is injective, $F(M)$ is injective by (3). From Theorem 1.7,it is sufficient to show that every submodule T of $F(M)$– projective module ${}_SU$ is $F(M)$–projective. Since $G({}_SU)$ is $GF(M)$–projective from (2), and $GF(M) \cong M$, so $G({}_SU)$ is M–projective. Since $G(T) \cong T' \leq G(U)$ from (1) and R is M–hereditary,$G(T)$ is M–projective. But $FG(T) \cong T$, so T is $F(M)$–prjective by (2).

The proof of the ($\Leftarrow$) part can be completed in a similar way.

2. Endommorphism Rings

In [3], the concept of relative hereditary module is given. A module ${}_RP$ is said to be M–hereditary in case that every submodule of ${}_RP$ is M–projective, M is a left R–module. P is said to be a self-hereditary in case that P is P–hereditary. And that the following two results are given in [3].

Proposition 2.1[3, Lemma 3.7] *If left R–module P is self- hereditary, then $P^{(n)}$ is self-hereditary for every natural number n.*

Proposition 2.2[3,Theorem 3.8] *If left R–module P is self- hereditary, then endommorphism ring end(${}_RP$) of module P is a left semi-hereditary ring .*

Theorem 2.3 *Let G be a generator of ${}_RM$, R be G–hereditary, I be a left ideal of R. Then*

(1) ${}_RI$ is left semi-hereditary;

(2) The ring $M_n(end({}_RI))$ of $n \times n$–matrices over $end({}_RI)$ is left semi-hereditary;

(3) For every idempotent element e of R, eRe is left semi-hereditary.

Proof (1) By Proposition 2.2, it is sufficient to show that ${}_RI$ is self-hereditary. We know from Proposition 17.6 in [1] that

$$G^{(n)} \cong R \oplus R',$$

where R' is a left $R-$module, n is a natural number. Let ${}_RL \leq {}_RI$, then L is left ideal of R. Therefore by the assumption that R is $G-$hereditary, L is $G-$projective. Hence by (2) in Proposition 1.4 , L is $R \oplus R'-$projective. And hance L is ${}_RI-$projective from (1) of Proposition 1.4, i.e., ${}_RI$ is self-herediatry.

(2) By (1) and Proposition 2.1, we have that ${}_RI^{(N)}$ is self- hereditary for every natural number, so $\mathrm{end}({}_RI^{(n)})$ is left semi-hereditary. But $\mathrm{end}({}_RI^{(N)}) \cong M_n(\mathrm{end}({}_RI))$ by Proposition 13.2 in [1], thus $M_n(\mathrm{END}({}_RI))$ is left semi-hereditary for every natural number.

(3) Since R is a left ideal of R for every idempotent element of R, $\mathrm{end}(R)$ is left semi-hereditary by (1). But $\mathrm{end}(R) \cong eRe$ by Proposition 5.9 in [1], thus eRe is left semi-hereditary.

In particular, R is left semi-hereditary. □

Theorem 2.4 *Let G be an injective generator of ${}_RM$, R be $G-$hereditary, M be a finitely generated $G-$projective left $R-$module. Then*

(1) $\mathrm{end}({}_RM)$ is left semi-hereditary;

(2) The matrix ring $M_n(\mathrm{end}({}_RM))$ over $\mathrm{end}({}_RM)$ is left semi-hereditary for every natural number.

Proof (1) It is sufficient to show that ${}_RM$ is self- hereditary by Proposition 2.2. Since M is finitely generated, there exists an epimophism $f : G^{(n)} \to M, n$ is a natural number. Let ${}_RN \leq {}_RM$. Since M is $G-$projective, N is $G-$projective by Theorem 1.7. Hence N is $G^{(n)}$-projective by (2) of Proposition 1.4, and hence N is $M-$projective by (1) of Proposition 1.4. Thus M is self-hereditary.

(2) The proof is similar to that of (2) of **Theorem 2.3** . □

References

[1] F.W.Anderson and K.R. Fuller, *Rings and Categories of Modules*, Springer-verlag, New York, 1973.

[2] C.Faith, *Algebra I,Rings, Modules, and Categories*, Springer-Verlag, Berlin, Heidelberg, 1981.

[3] Tang Zhongming, *Relative herditary modules*, J. Mathematical Research & Exposition, 1(1988), 1-5.

相　对　遗　传　环

郭　宇

(长春师范学院数学系，130032)

本文给出了左相对遗传环的概念及刻划．讨论了左相对遗传环与左遗传环，左半遗传环的关系，并给出了左遗传环的几个新的刻划．最后，证明了左相对遗传环（相对于左模范畴的生成元）的左理想的自同态环是左半遗传的；左相对遗传环（相对于左模范畴的内射生成元）上的有限生成相对投射模的自同态环是左半遗传的等结果．