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An Extended Schlömilch Formula and Its Applications*

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Let $\{A_1(n,k), A_2(n,k)\}$ be a Stirling-type pair generated by the formal expansions

$$\frac{(f(t))^k}{k!} = \sum_{n\geq 0} A_1(n,k)\frac{t^n}{n!}, \qquad \frac{(g(t))^k}{k!} = \sum_{n\geq 0} A_2(n,k)\frac{t^n}{n!}, \tag{1}$$

where $f(t) = \sum a_k t^k$ and $g(t) = \sum b_k t^k$ are reciprocal formal power series over the complex field with $a_0 = b_0 = 0$ and $a_1 = b_1 = 1$, namely $f(g(t)) = g(f(t)) = t$. Then we have a general Schlömilch formula

$$A_i(n,k) = \sum_{r=0}^{n-k} (-1)^r \binom{2n-k}{n-k-r}\binom{n-1+r}{n-k+r} A_j(n-k+r,r), \tag{2}$$

where the index pair (i,j) may be either of $(1,2)$ and $(2,1)$, so that the formula expresses a kind of reciprocal relation between the two kinds of generalized Stirling numbers.

The formula (2) can be proved by using Lagrange's inversion theorem and a certain differentiation formula for powers of functions.

Applications

(a) Certain asymptotic expansions of $A_i(n,k)$, $(i = 1,2)$ can be derived from (2) for the case $(n,k) \longrightarrow (\infty,\infty)$ with $n-k =$ constant.

(b) It may be shown via (2) that the central factorial numbers of the first and second kind $t_1(n,k)$ and $t_2(n,k)$ have sum-expressions of rank 2 and of rank 1, repectively.

(c) The following theorem can be proved by means of (2): Suppose that $A_1(n,k)$ as well as $A_2(n,k)$ are all integers. Let p be an odd prime number and k be any integer such that $1 < k < p < 2k-1$. Then the following congruence relations

$$A_i(p+j, k+j) \equiv 0 \pmod p, \quad (i = 1,2) \tag{3}$$

hold for all j $(0 \leq j \leq p-k)$. Details of the above-mentioned results and related ones will be published elsewhere.

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