Journal of Mathematical Research & Exposition,
Vol.12, No.2, May 1992

A Note on the Number of Loopless Eulerian Planar Maps*

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Abstract. This paper provides the number of combinatorially distinct rooted loopless Eulerian planar maps with given edge number. Meanwhile, an explicit formula for the almost loopless case is presented.

In the paper "On the vertex partition equation of loopless Eulerian planar maps" [1], we found the cubic equation satisfied by the generating function of rooted loopless Eulerian planar maps with the edge number and the valency of the root-vertex as parameters as

$$\begin{aligned}&x^4y^2f^3+2(1-x^2y)x^2yf^2\\&+(1-x^2-x^2y+(1+f^*)x^4y^2-2x^2yf^*-x^2y^2(f^*)^2)f\\&=1-x^2-x^2yf^*+x^2y^2(1-f^*)f^*,\end{aligned}\tag{1}$$

where $f^*=f(1,y)$.

No doubt, it is difficult to solve (1). However, in this note, we provide an indirect way to find f^* and then f from Eg.(1).

First, we recall the parametric expressions for the general Eulerian case. Let $h(y)$ be the generating function of rooted Eulerian planar maps with the edge number as the parameter. We know that y and h satisfy the following parametric expressions in [2]:

$$\begin{cases} y=-\frac{(\xi-1)(\xi-2)}{\xi^2};\\ h=\frac{1+\xi-\xi^2}{(\xi-2)^2}.\end{cases}\tag{2}$$

Now, we introduce some kinds of new maps. An Eulerian planar map is said to be a bounded loop map whenever all the edges on the boundary of the root-face are loops. A boundary loop map is called an inner map if it has a single edge on the boundary of the root-face. Suppose h_{BL}, h_{in} are the generating functions of rooted boundary loop maps and rooted inner maps respectively. Then we have

$$\begin{cases} h_{in}=yh;\\ h_{BL}=\frac{h_{in}}{1-h_{in}}.\end{cases}\tag{3}$$

*Received June 4, 1990. Research supported by the National Natural Science Foundation of China.

Here, we have to notice that the vertex map is neither a boundary loop map nor an inner map. From (3), we have

$$h_{BL} = \frac{yh}{1 - yh}. \tag{4}$$

For conveneince, we call the map whose valency of the rooted vertex is 2 to be a 2-root-valency map. Let $\tilde{h_{nl}}$ and $\tilde{h}$ be the generating functions of 2-root-valency loopless and general Eulerian planar maps with the edge number as the parameter, respectively. By the same procedure as shown in [3,4], we may see that

$$(1 + h_{BL})^2(\tilde{h} - y) = \tilde{h_{nl}}(y(1 + h_{BL})^2). \tag{5}$$

Let $z = y(1 + h_{BL})^2$. Then from (3) and (4), we have

$$\begin{cases} z = y(1 - yh))^{-2}; \\ \tilde{h_{nl}}(z) = (1 - yh)^{-2}(\tilde{h} - y). \end{cases} \tag{6}$$

Whenever noticing that for any Eulerian planar map except for the vertex map, which is assumed to be Eulerian, we may get a 2-root-valency one by bisectioning the root-dege.Conversely, from a 2-root-valency Eulerian map, we may also get an Eulerian one, which must not be the vertex map, by combining the two edges incident with the root-vertex into one. The loop map which is a 2-root-valency map can also be treated as the resultant map of adding a loop on the vertex map. Therefore, we have

$$\tilde{h} = yh. \tag{7}$$

According to (2), (6) and (7), we may finally obtain that

$$\begin{cases} z = -\xi^2(\xi - 1)(\xi - 2)^3; \\ \tilde{h_{nl}} = -z\frac{(\xi-1)(2\xi-3)}{(\xi-2)^3}. \end{cases} \tag{8}$$

By using the similar procedure from which (7) is derived, we have

$$\tilde{h_{nl}} = yh_{al},$$

where h_{al} is the generating function of rooted almost loopless Eulerian planar maps with the edge number as the parameter, in which only the root-edge is allowed to be a loop. But, the vertex map is not defined to be an almost loopless map here.

From (8), we have

$$\begin{cases} z = \xi^2(1 - \xi)(\xi - 2)^3; \\ h_{al} = \frac{(\xi-1)(2\xi-3)}{(\xi-2)^3}. \end{cases} \tag{9}$$

On the basis of (9), by employing the Lagrangian inversion[5], we find that the coefficient of z^n, $n \geq 1$, is

$$\sum_{i=0}^{n-1}(-1)^i \frac{(2n + 4i - 1)}{n(2n - 1)} \binom{4n - i + 1}{3n + 2} \binom{2n + i - 2}{2n - 2}.$$

Of course, the constant term of h_{al} must be zero. This is the number of combinatorially distinct rooted almost loopless Eulerian planar maps with n edges.

Moreover, it can be verified that the following identities are valid:

$$\sum_{i=0}^{n-1}(-1)^i\binom{4n-i+1}{3n+2}\binom{2n+i-2}{2n-2}=\sum_{j=0}^{[\frac{n-1}{2}]}\binom{2n+2-2j}{n-1-2j}\binom{2n-2-j}{j};\tag{10}$$

$$\sum_{i=0}^{n-1}(-1)^i i\binom{4n-i+1}{3n+2}\binom{2n+i-2}{2n-2}=-(2n-1)\sum_{j=0}^{[\frac{n-2}{2}]}\binom{2n+j-1}{j}\binom{2n-2j}{n-2j-2}.\tag{11}$$

From the two identities (10) and (11), we may deduce that the number of combinatorially distinct rooted almost loopless Eulerian planar maps with n edges is

$$\frac{1}{n}\sum_{j=0}^{[\frac{n-1}{2}]}\binom{2n+2-2j}{n-1-2j}\binom{2n-2-j}{j}-\frac{4}{n}\sum_{j=0}^{[\frac{n-1}{2}]}\binom{2n+j-1}{j}\binom{2n-2j}{n-2j-2}\tag{12}$$

In this formula, both summations are over the positive terms.

Finally, we inverstigate h_{nl}, the generating function of rooted loopless Eulerian planar maps with the edge number as the parameter. In this case, the vertex map is defined to be include. This means that the constant term of h_{nl} is 1. It is easy to see that

$$h_{al}^*=yh_{al}^2,\tag{13}$$

where h_{al}^* is the generating function of the rooted almost loopless Eulerian planar maps with the edge number as the parameter in which the root- edges are loops.Since

$$h_{nl}=h_{al}-h_{al}^*,\tag{14}$$

from (13), we find the following equation:

$$h_{nl}-1=h_{al}-zh_{nl}^2.\tag{15}$$

Because h_{al} has been determined by (12), we may recursively extract h_{nl} term by term without difficulty according to the relations obtained by identifying the coefficients on both sides of Eq.(15). In this way, we may obtain

$$h_{nl}=1+y^2+y^3+6y^4+14y^5+\cdots,\tag{16}$$

which represents $h_{nl}=f^*$ appearing in Eq.(1).

However, we would like to find an explicit expression of h_{nl}. From (9) and (15), we have

$$\begin{cases} z=\xi^2(1-\xi)(\xi-2)^3;\\ h_{nl}=\frac{1}{2z}(-1+\sqrt{1-4z\frac{\xi^2-\xi-1}{(2-\xi)^2}}).\end{cases}\tag{17}$$

On the basis of the second term of (17), by expanding the square root into power series, we obtain

$$h_{nl}=\sum_{n\geq 0}\frac{(2n)!}{n!(n+1)!}\frac{(\xi^2-\xi-1)^{n+1}}{(2-\xi)^{2n+2}}z^n.\tag{18}$$

In principle, we may express $(\xi^2-\xi-1)^{n+1}(2-\xi)^{-(2n+2)}$ as a power series of z by the Lagarngian inversion. Because of the complicatedness involved, we are here not allowed to spend too much space to do it. It would be very interesting to hit upon a way for the sake of seeking a simple formula of $f^*=h_{nl}$, and then of f from Eq.(1).

We now provide another way to determine h_{nl}. First, we need to introduce some kinds of maps. A map is called loop rooted map if in which the root-edges are loops. Let h_l be the generating function of loop rooted Eulerian planar maps with the edge number as the parameter. It is easy to show that

$$h_l=yh^2. \tag{19}$$

A map is called link rooted map if in which the root-edges are not loops. Let h_r be the corresponding function of link rooted Eulerian planar maps. We have

$$h_r=h-h_l, \tag{20}$$

and the vertex map is defined to be included in the set of link rooted ones.

Furthermore, we may also see that

$$h_r=h_{nl}(y(1+h_{BL})^2), \tag{21}$$

in which, the contribution of the vertex map to the both sides is considered as a degenerate case.

From(2), (4), and (19–21), we can derive that

$$\begin{cases} z=-\xi^2(\xi-1)(\xi-2)^3; \\ h_{nl}=\frac{1+\xi-\xi^2}{\xi^2(2-\xi)^3}. \end{cases} \tag{22}$$

Thus, we are now allowed to apply the Lagrangian inversion to determine h_{nl} as a power series of z.

The coefficient of z^n in h_{nl}, the number of rooted loopless Eulerian planar maps with n edges, is

$$A_n=\frac{1}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\xi^{n-1}}\left(\frac{(1-\xi^2)(3\xi-4)}{\xi^{2n+3}(2-\xi)^{3n+4}}\right)_{\xi=1}=\frac{(n-1)!}{n!}\frac{\mathrm{d}^{n-2}}{\mathrm{d}\xi^{n-2}}\left(\frac{4+\xi-3\xi^2}{\xi^{2n+3}(2-\xi)^{3n+4}}\right)_{\xi=1}.$$

Write

$$\begin{aligned} S_l &= \frac{1}{(n-2)!}\frac{\mathrm{d}^{n-2}}{\mathrm{d}\xi^{n-2}}\left(\frac{\xi^l}{\xi^{2n+3}(2-\xi)^{3n+4}}\right)_{\xi=1} \\ &= \sum_{i=0}^{n-2}(-1)^i\binom{2n+2+i-l}{i}\binom{4n+1-i}{n-i-2} \end{aligned} \tag{23}$$

for $l=0,1,$ and 2. Then we have

$$A_n=\frac{1}{n}(aS_0+S_1-3S_2). \tag{24}$$

It is not difficult to verify the following identities:

$$\sum_{i=0}^{n-2}(-1)^i \binom{2n+2+i-l}{i}\binom{4n+1-i}{n-i-2}$$
$$= \sum_{j=0}^{[\frac{n-2}{2}]} \binom{2n+2-l+j}{j}\binom{2n-2+l-2j}{n-2-2j} \tag{25}$$

for $n \geq 2, -(n+1) \leq l < 2n+3$. From (25), for $l = 0, 1, 2$, we have

$$S_l = \sum_{j=0}^{[\frac{n-2}{2}]} \binom{2n+2-l+j}{j}\binom{2n-2+l-2j}{n-2-2j}, \tag{26}$$

all of which are expressed as summation of positive terms.

In consequence, by substuting (26), for $l = 0, 1$, and 2, into (24), we may find

$$A_n = 3\sum_{j=0}^{[\frac{n-2}{2}]} \frac{(2n+1-j)(4j-n+4)-2}{n(n+1)(n+2)} \times \binom{2n+j}{j}\binom{2n-2-2j}{n}, \tag{27}$$

for $n \geq 1$. Of course, $A_0 = 1$, the constant term of h_{nl} which corresponds to the vertex map, the degenerate case.

Finally, we point out that it seems to be possible to establish a quadratic equatiion instead of Eq.(1), the cubic one.

References

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关于无环 Euler 平面地图数目的注记

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摘 要

本文提供了组合上不等价的有根无环 Euler 平面地图以边数为参数的的数目,同时对于几乎无环的情形也给出了一个计数显式.