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A Note on Stability of Multivariate Distributions*

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Abstract. In this note, we give an elementary proof of a characterization for stability of multivariate distributions by considering a functional equation.

1. Introduction

A probability measure μ on $\mathbf{R}^d$ is said to be stable if there are sequences $\{X_n\}$, $\{a_n\}$, and $\{b_n\}$ such that $\{X_n\}$ are independent and identically distributed $\mathbf{R}^d-$ valued random vectors, $a_n \in \mathbf{R}^d, b_n > 0$, and the distribution of $b_n^{-1}\sum_{j=1}^{n} X_j - a_n$ converges to μ. As is well known, μ is stable if and only if, for each $c_1 > 0$ and $c_2 > 0$, there exist $c > 0$ and $a \in \mathbf{R}^d$ such that the characteristic function $\hat{\mu}$ of μ satisfies

$$\hat{\mu}(c_1 t)\hat{\mu}(c_2 t) = \hat{\mu}(ct)e^{ia't}, \ \forall t \in \mathbf{R}^d. \tag{1}$$

In this case, c is uniquely determined by $c = (c_1^\alpha + c_2^\alpha)^{1/\alpha}$, with the characteristic exponent $0 < \alpha \le 2$, independent of c_1 and c_2.

It is natural to ask what if we only assume (1) is satisfies by a single collection of c_1, c_2, c and a. The purpose of this note is to present an elementary proof of the answer: in most of the cases, this much weaker condition characterizes the stability of μ. Our approach is to reduce the characterization problem to solving a functional equation. The main theorem in this note is applied in Zeng [5] to characterize multivariate stable distributions via random linear statistics.

2. The main theorem

The main result of this note is

Theorem 1 *A probability measure μ on $\mathbf{R}^d$ is stable if and only if there exist positive numbers c_1, c_2, c and $a \in \mathbf{R}^d$ such that c_1/c and c_2/c are non-commensurable (i.e., there are no integers m and n such that $(c_1/c)^m = (c_2/c)^n$), and the equation (1) holds.*

We first prove a lemma.

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Lemma 1 *Let $f:(0,\infty)\to\mathbf{R}$ be a monotone function, and let $a,b\in(0,1)$ be non-commensurable. If*

$$f(ax)+f(bx)=f(x),\quad \forall x>0, \tag{2}$$

then $f(x)=cx^{\alpha}$, where $a^{\alpha}+b^{\alpha}=1$, and c is a constant.

Proof Without loss of generality, we can assume f is nondecreasing, not identically zero, $a<b$, and $a+b=1$ (i.e.,$\alpha=1$).

Let $c=\inf_{x>0}(f(x)/x)$, then $c\neq 0$. For any $\delta>0$, it is clear that if $kx\leq f(x)\leq Kx$ on $(a\delta,\delta]$ for some k and K, then the same inequalities hold on $(a\delta,\infty)$. This implies that

$$c=\inf_{0<x\leq\delta}\left(\frac{f(x)}{x}\right), \tag{3}$$

where δ is any positive number. We will show that for any $\varepsilon>0, f(x)/x<c+\varepsilon$ for all $x>0$, so that the lemma follows.

Since a and b are non-commensurable, the set $\{a^m b^n : m,n\in\mathbf{N}\}$ can be indexed as a decreasing sequence $1=a_0>a_1>a_2>\cdots$, and $\lim_{n\to\infty}(\frac{a_n}{a_{n+1}})=1$. Let $\varepsilon_1=\frac{\varepsilon}{2|c|}$ (since $c\neq 0$), then there exists n_0 such that

$$\frac{a_n}{a_{n+1}}<1+\varepsilon_1,\quad \text{for all}\quad n\geq n_0. \tag{4}$$

In light of (3), we can also find $x_0<\delta$ with

$$\frac{f(x_0)}{x_0}<c+\varepsilon_2,$$

where $\varepsilon_2=aa_{n_0}\frac{\varepsilon}{2}$. By applying the equation (2) repeatedly, we have

$$f(x_0)=f(a_nx_0)+\sum_{j\in I}f(a_jx_0),$$

where $I\subset N$ is a finite set, $a_n+\sum_{j\in I}a_j=1$, and $n\geq n_0$. It follows that

$$f(a_nx_0)=f(x_0)-\sum_{j\in I}f(a_jx_0),$$

and hence

$$f(a_nx_0)<cx_0+\varepsilon_2x_0-\sum_{j\in I}ca_jx_0=c(a_nx_0)+\varepsilon_2x_0, \tag{5}$$

for all $n\geq n_0$. Let $t\in(aa_{n_0}x_0,a_{n_0}x_0]$, then $t\in(a_{n+1}x_0,a_nx_0]$ for some $n\geq n_0$. The monotonicity of f along with (4) and (5) yield that

$$f(t)\leq f(a_nx_0)<c(a_nx_0)+\varepsilon_2x_0<ct+\varepsilon t,$$

i.e.,

$$\frac{f(t)}{t}<c+\varepsilon,\quad \forall t\in(aa_{n_0}x_0,a_{n_0}x_0],$$

and the inequality is valid for all $x > 0$.

Remark If we assume f is nonnegative, Lemma 1 becomes a special case of the Lau-Rao's theorem (1982) on the integrated Cauchy functional equation. We give here a direct simple proof, inspired by Shanbhag (1977), without the nonnegativeness assumption. Following the same lines with slight modification, we can show tha the same conclusion holds for a more general functional equation

$$f(x) = \sum_{j=1}^{n} d_j f(a_j x),$$

where the a_j's are not commensurable, and $\sum_{j=1}^{n} d_j a_j^{\alpha} = 1$.

Proof of Theorem 1 Let μ be a probability measure on $\mathbf{R}^d$, $\hat{\mu}$ its characteristic function. If μ is stable with characteristic exponent α, then for any $c_1 > 0$ and $c_2 > 0$, there exist $c > 0$ and $a \in \mathbf{R}^d$ such that the characteristic function $\hat{\mu}$ of μ satisfies the equation (1) and $c^{\alpha} = c_1^{\alpha} + c_2^{\alpha}$. This equation on c's can be written as $(c_1/c)^{\alpha} + (c_2/c)^{\alpha} = 1$. To show the existence of non- commensurable c_1/c and c_2/c, we observe the function $\phi(x) = \frac{\log(1-x^{\alpha})}{\alpha \log x}$ defined on $(0,1)$. The continuity of ϕ on the interval $(0,1)$ implies that for almost all of the pairs c_1/c and c_2/c,$\log(c_1/c)/\log(c_2/c)$ is irrational, and hence the necessity follows.

Conversely, let c_1, c_2, c and a be such that

$$\hat{\mu}(c_1 t)\hat{\mu}(c_2 t) = \hat{\mu}(ct)e^{ia't}, \quad \forall t \in \mathbf{R^d},$$

and $\beta_1 = c_1/c$ and $\beta_2 = c_2/c$ are non-commensurable. Then $c \geq c_j, j = 1, 2$, with the equality only in trivial case, and hence equation (1) can be rewritten as

$$\hat{\mu}(\beta_1 t)\hat{\mu}(\beta_2 t) = \hat{\mu}(t)e^{i\gamma' t}, \quad \forall t \in \mathbf{R^d}. \tag{6}$$

where $0 < \beta_1, \beta_2 < 1$, excluding the trivial case, and $\gamma \in \mathbf{R^d}$. Notice that the equation (6) also implies that μ is infinitely divisible, the Lévy canonical representation gives

$$\hat{\mu}(t) = \exp\{iP_1(t) + P_2(t) + \int_{\mathbf{R}^d} \left(e^{iw't} - 1 - \frac{iw't}{1 + |w|^2}\right) \mathrm{d}V(w)\},$$

where P_1 and P_2 are homogeneous polynomials of degree one and two, respectively, V is a measure on $\mathbf{R}^d$ such that $V(\{0\}) = 0$ and

$$\int_{\mathbf{R}^d} \left(\frac{|w|^2}{1 + |w|^2}\right) \mathrm{d}V(w) < \infty. \tag{7}$$

Let α be the unique positive number determined by $\beta_1^{\alpha} + \beta_2^{\alpha} = 1$. Then the spectral measure V satisfies

$$\mathrm{d}V(\frac{w}{\beta_1}) + \mathrm{d}V(\frac{w}{\beta_2}) = \mathrm{d}V(w). \tag{8}$$

For Borel sets $E \subseteq \mathbf{R}_+$ and $B \subseteq \mathbf{S}^{d-1}$ (the unit sphere in $\mathbf{R}^d$), let EB denote the set of w such that $w = su, s \in E, u \in B$. The equation (8) implies that

$$\int_{EB} \mathrm{d}V(\frac{w}{\beta_1}) + \int_{EB} \mathrm{d}V(\frac{w}{\beta_2}) = \int_{EB} \mathrm{d}V(w).$$

Given $B \subseteq \mathbf{S}^{d-1}$, if we let $N(x,B)=\int_{(x,\infty)B} \mathrm{d}V(w)$, then $N(x,B)$ is a momotone decreasing function on $(0,\infty)$, satisfying

$$N(\frac{x}{\beta_1},B)+N(\frac{x}{\beta_2},B)=B(x,B), \quad \forall x>0, \tag{9}$$

Since β_1 and β_2 are non-commensurable, it follows from Lemma 1 that $N(x,B)=c(B)x^{-\alpha}$, for any Borel set $B \subseteq \mathbf{S}^{d-1}$, and the spectral measure V should be of the form

$$V(EB)=\int_{EB} \frac{\mathrm{d}s\mathrm{d}\Phi(u)}{s^{\alpha+1}}, \quad \text{for } E \subseteq \mathbf{R}_+, B \subseteq \mathbf{S}^{d-1}, \tag{10}$$

where Φ is a positive finite measure on the unit sphere $\mathbf{S}^{d-1}$. Hence, the equation (1) together with (7) and (10) yield that $0<\alpha \leq 2$, unless μ is degenerate, and the spectral measure V is zero if $\alpha=2$, in which case μ is a normal law, or otherwise, V is given by (10), and the quadratic form $P_2=0$, so that μ is a stable law with characteristic exponent $0<\alpha<2$. □

3. Further characterizations of stable laws

The following characterizations of multivariate stable distributions through identically distributed linear statistics (e.g.,see Gupta et al) can be easily derived from Theorem 1.

Theorem 2 *Let X_1, X_2 and X_3 be independent and identically distributed random vectors in $\mathbf{R}^d$. Then X_1 has a multivariate stable distribution if and only if there exists α, $0<\alpha \leq 2$, and $a_1, a_2 \in \mathbf{R}^d$, such that $2^{1/\alpha}X_1+a_1$ and $X_1+X_2, 3^{1/\alpha}X_1+a_2$ and $X_1+X_2+X_3$ are identically distributed, respectively.*

Proof Let $\hat{\mu}$ denote the characteristic function of X_1. The given conditions imply that $X_1+a, a \in \mathbf{R}^d$, is identically distributed with $(\frac{2}{3})^{1/\alpha}X_1+(\frac{1}{3})^{1/\alpha}X_2$, and we have

$$\hat{\mu}\left((\frac{2}{3})^{1/\alpha}t\right)\hat{\mu}\left((\frac{1}{3})^{1/\alpha}t\right)=\hat{\mu}(t)e^{ia't}, \quad \forall t \in \mathbf{R}^d.$$

Since $(\frac{2}{3})^{1/\alpha}$ and $(\frac{1}{3})^{1/\alpha}$ are not commensurable, Theorem 1 implies that $\hat{\mu}$ is stable with characteristic exponent α. □

We conculde with two special cases when $\alpha=1$ and $\alpha=2$.

Corollary *Let X_1, X_2 and X_3 be independent and identically distributted random vectors in $\mathbf{R}^d$. Then*

(1). *X_1 has a multivariate stable distribution with Cauchy marginals if $2X_1$ and $X_1+X_2, 3X_1$ and $X_1+X_2+X_3$ are identically distributed, respectively.*

(2). *X_1 has a multivariate normal distribution (possibly degenerate) with zero location vector if and only if $\sqrt{2}X_1$, and $X_1+X_2, \sqrt{3}X_1$ and $X_1+X_2+X_3$ are identically distributed, respectively.*

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