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A Note on the Toughness of Generalized Petersen Graphs*

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Abstract. We show that, for each integer $n \geq 5$, the toughness of a generalized Petersen graph with $2n$ vertices is less than or equal to 4/3, and 4/3 is the best upper bound.

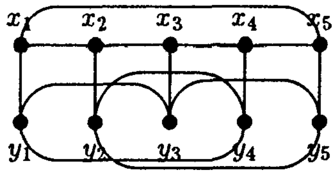
Let G be a graph and $V(G)$ and $E(G)$ be its vertex-set and edge-set respectively. The toughness of G is defined as

$$t(G) = \min\{\frac{|S|}{\omega(G-S)}\},$$

where the mininum is taken over all disconnecting set S of $V(G)$, $|S|$ is the cordinality of S, and $\omega(G-S)$ is the number of components of the induced graph $G-S$. Let n be an integer ≥ 5. A generalized Petersen graph $G(n,2)$ is a graph with

$$\begin{aligned} &V(G(n,2)) = \{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n\}, \quad \text{and} \\ &E(G(n,2)) = \{(x_i, x_{i+1}), (y_i, y_{i+2}), (x_i, y_i); \\ &i = 1, 2, \cdots, n, \quad \text{and the subscripts are taken modulo } n\}. \end{aligned}$$

Thus, $G(5,2)$ is the Petersen graph which is drawn as follows:



A conjecture in [1] stated that if $n \geq 5$ and $n \equiv 1 \bmod 4$, then $t(G(n,2)) = 4/3$. Here, we show that the conjecture does not hold in general. In fact, motivated by the conjecture, we shall prove the following:

Theorem *Let n be an integer ≥ 5. Then $t(G(n,2)) \leq 4/3$.*

Example Consider $G(13,2)$ with $V(G(13,2)) = \{x_1, x_2, \cdots, x_{13}, y_1, y_2, \cdots, y_{13}\}$. Let $S = \{x_4, x_7, x_{11}, x_{13}, y_2, y_5, y_6, y_9, y_{12}\}$ be a disconnecting set in $G(13,2)$. Then $G(13,2)-S$ is the following graph:

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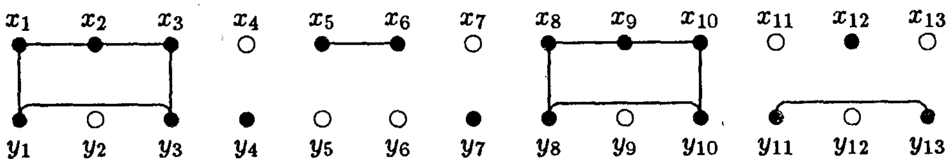


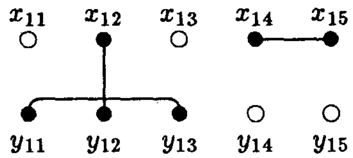
Thus, $|S| = 9$ and the components in $G(13,2) - S$ are the induced graphs of the following sets:

$$\{y_1, x_1, x_2, x_3, y_3\}, \{y_4\}, \{x_5, x_6\}, \{y_7\}, \{y_8, x_8, x_9, x_{10}, y_{10}\}, \{x_{12}\}, \text{ and } \{y_{11}, y_{13}\}.$$

Consequently $t(G(13,2)) \leq |S|/\omega(G(13,2) - S) = 9/7 < 4/3$.

The proof of the theorem goes as follows:

Case 1. $n \equiv 0 \bmod 5$, i.e., $n = 5k$ where k is an integer ≥ 1. For $k = 1$, let $V(G(5,2)) = \{x_{1i}, y_{1i}, i = 1, 2, \cdots, 5\}$, and $S = \{x_{11}, x_{13}, y_{14}, y_{15}\}$ be a disconnecting set in $G(5,2)$. Then $G(5,2) - S$ is the following graph:



Thus, $|S| = 4$ and the components in $G(5,2) - S$ are the induced graphs of the following sets:

$$\{x_{12}, y_{12}\}, \{y_{11}, y_{13}\}, \text{ and } \{x_{14}, x_{15}\}.$$

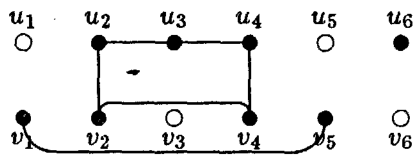
Consequently $t(G(5,2)) \leq |S|/\omega(G(5,2) - S) = 4/3$.

For $k > 1$, we "patch" k copies of $G(5,2)$ graphs together. Let $V(G(5k,2)) = \{x_{ji}, y_{ji}, i = 1, 2, \cdots, 5 \text{ and } j = 1, 2, \cdots, k\}$, and $S = \{x_{j1}, x_{j3}, y_{j4}, y_{j5}, j = 1, 2, \cdots, k\}$ be a disconnecting set in $G(5k,2)$. Then the components in $G(5k,2) - S$ are the induced graphs of the following sets:

For $j = 1, 2, \cdots, k, \{x_{j2}, y_{j2}\}, \{y_{j1}, y_{j3}\}$, and $\{x_{j4}, x_{j5}\}$. Thus, $|S| = 4k, \omega(G(5k,2) - S) = 3k$, and $t(G(5k,2)) \leq |S|/\omega(G(5k,2) - S) = (4k)/(3k) = 4/3$.

Case 2. $n \equiv 1 \bmod 5$, i.e., $n = 6 + 5k$ where k is an integer ≥ 0.

For $k = 0$, let $V(G(6,2)) = \{u_m, v_m, m = 1, 2, \cdots, 6\}$ and $S = \{u_1, u_5, v_3, v_6\}$ be a disconnecting set in $G(6,2)$. Then $G(6,2) - S$ is the following graph:



Thus, $|S| = 4$, and the components in $G(6,2) - S$ are the induced graphs of the following sets:

$$\{v_1, v_5\}, \{v_2, u_2, u_3, u_4, v_4\} \text{ and } \{u_6\}.$$

Consequently $t(G(6,2)) \leq |S|/\omega(G(6,2)-S) = 4/3$.

For $k \geq 1$, we "patch" $G(6,2)$ with k copies of $G(5,2)$ graphs. Let

$$V(G(6+5k,2)) = \{u_m, v_m, x_{ji}, y_{ji}, m = 1,2,\cdots,6, i = 1,2,\cdots,5 \text{ and } j = 1,2,\cdots,k\},$$

and

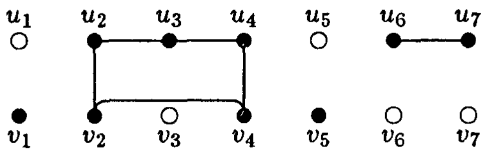
$$S = \{u_1, u_5, v_3, v_6, x_{j1}, x_{j3}, y_{j4}, y_{j5}, j = 1,2,\cdots,k\}$$

be a disconnecting set in $G(6+5k,2)$. Then the components in $G(6+5k,2)-S$ are the induced graphs of the following sets: $\{v_1\}, \{v_2,u_2,u_3,u_4,v_4\}, \{u_6\}, \{v_5,y_{11},y_{13}\}$, $\{x_{12},y_{12}\}, \{x_{14},x_{15}\}$, and for $j = 2,3,\cdots,k, \{x_{j2},y_{j2}\}, \{y_{j1}, yj3\}$, and $\{x_{j4},x_{j5}\}$.

Thus, $|S| = 4+4k, \omega(G(6+5k,2)-S) = 6+3(k-1)$, and $t(G(6+5k,2) \leq |S|/\omega(G(6+5k,2)-S) = (4+4k)/(6+3(k-1)) = 4/3$.

Case 3. $n \equiv 2 \bmod 5$, i.e., $n = 7+5k$ where k is an integer ≥ 0.

For $k = 0$, let $V(G(7,2)) = \{u_m, v_m; m = 1,2,\cdots,7\}$, and $S = \{u_1,u_5,v_3,v_6,v_7\}$ be a disconnecting set in $G(7,2)$. Then $G(7,2)-S$ is the following graph:



Thus, $|S| = 5$, and the components in $G(7,2)-S$ are the induced graphs of the following sets:

$$\{v_1\}, \{v_2,u_2,u_3,u_4,v_4\}, \{v_5\}, \text{ and } \{u_6,u_7\}.$$

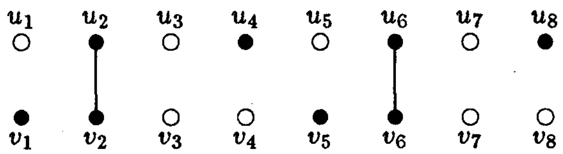
Consequently $t(G(7,2)) \leq |S|/\omega(G(7,2)-S) = 5/4 < 4/3$.

For $k \geq 1$, we "patch" $G(7,2)$ with k copies of $G(5,2)$ graphs. Let $V(G(7+5k,2)) = \{u_m, v_m, x_{ji}, y_{ji}; m = 1,2,\cdots,7, i = 1,2,\cdots,5 \text{ and } j = 1,2,\cdots k\}$, and $S = \{u_1,u_5,v_3,v_6,v_7,x_{j1},x_{j3},y_{j4},y_{j5}; j = 1,2,\cdots,k\}$ be a disconnecting set in $G(7+5k,2)$. Then the components in $G(7+5k,2)-S$ are the induced graphs of the following sets: $\{v_1\}, \{v_2,u_2,u_3,u_4,v_4\}$, $\{v_5\}, \{u_6,u_7\}$, and for $j = 1,2,\cdots,k, \{x_{j2},y_{j2}\}, \{y_{j1},y_{j3}\}$, and $\{x_{j4},x_{j5}\}$.

Thus, $|S| = 5+4k, \omega(G(7+5k,2)-S) = 4+3k$, and $t(G(7+5k,2)) \leq |S|/\omega(G(7+5k,2)-S) = (5+4k)/(4+3k) < 4/3$.

Case 4. $n \equiv 3 \bmod 5$, i.e., $n = 8+5k$ where k is an integer ≥ 0.

For $k = 0$, let $V(G(8,2)) = \{u_m, v_m; m = 1,2,\cdots,8\}$ and $S = \{u_1,u_3,u_5,u_7,v_3,v_4,,v_7,$ $v_8\}$ be a disconnecting set in $G(8,2)$. Then $G(8,2)-S$ is the following graph:



Thus, $|S| = 8$, and the components in $G(7,2) - S$ are the induced graphs of the following sets: $\{v_1\}, \{v_2, u_2\}, \{u_4\}, \{v_5\}, \{v_6, u_6\}$ and $\{u_8\}$.

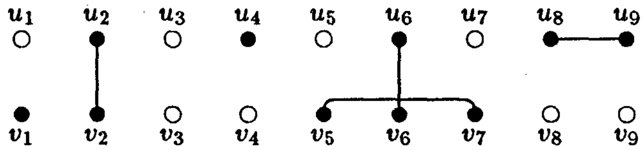
Consequently $t(G(8,2)) \leq |S|/\omega(G(8,2) - S) = 8/6 = 4/3$.

For $k \geq 1$, we "patch" $G(8,2)$ with k copies of $G(5,2)$ graphs. Let $V(G(8+5k,2)) = \{u_m, v_m, x_{ji}, y_{ji}; m = 1,2,\cdots,8, i = 1,2,\cdots,5$ and $j = 1,2,\cdots,k\}$, and $S = \{u_1, u_3, u_5, u_7, v_3, v_4, v_7, v_8, x_{j1}, x_{j3}, y_{j4}, y_{j5}; j = 1,2,\cdots,k\}$ be a disconnecting set in $G(8+5k,2)$. Then the components in $G(8+5k,2) - S$ are the induced graphs of the following sets: $\{v_1\}, \{v_2, u_2\}, \{u_4\}, \{v_5,\}, \{v_6, u_6\}, \{u_8\}$ and for $j = 1,2,\cdots,k, \{x_{j1}, y_{j2}\}, \{y_{j1}, y_{j3}\}$ and $\{x_{j4}, x_{j5}\}$.

Thus, $|S| = 8+4k$ and $\omega(G(8+5k,2) - S) = 6+3k$, and $t(G(8+5k,2)) \leq |S|/\omega(G(8+5k,2) - S) = (8+4k)/(6+3k) = 4/3$.

Case 5. $n \equiv 4 \bmod 5$, i.e., $n = 9+5k$ where k is an integer ≥ 0.

For $k = 0$, let $V(G(9,2)) = \{u_m, v_m; m = 1,2,\cdots 9\}$ and $S = \{u_1, u_3, u_5, u_7, v_3, v_4, v_8, v_9\}$ be a disconnecting set in $G(9,2)$. Then $G(8,2) - S$ is the following graph:



Thus, $|S| = 8$, and the components in $G(9,2) - S$ are the induced graphs of the following sets: $\{v_1\}, \{v_2, u_2\}, \{u_4\}, \{v_5, v_7\}, \{v_6, u_6\}$ and $\{u_8, u_9\}$.

Consequently $t(G(9,2)) \leq |S|/\omega(G(9,2) - S) = 8/6 = 4/3$.

For $k \geq 1$, we "patch" $G(9,2)$ with k copies of $G(5,2)$ graphs. Let $V(G(9+5k,2)) = \{u_m, v_m, x_{ji}, y_{ji}; m = 1,2,\cdots,9, i = 1,2,\cdots,5$ and $j = 1,2,\cdots,k\}$, and $S = \{u_1, u_3, u_5, u_7, v_3, v_4, v_8, v_9, x_{j1}, x_{j3}, y_{j4}, y_{j5}; j = 1,2,\cdots,k\}$ be a disconnecting set in $G(9+5k,2)$. Then the components in $G(9+5k,2) - S$ are the induced graphs of the following sets: $\{v_1\}, \{v_2, u_2\}, \{u_4\}, \{v_5, v_7\}, \{u_8, u_9\}$, and for $j = 1,2,\cdots,k, \{x_{j2}, y_{j2}\}, \{y_{j1}, y_{j3}\}$ and $\{x_{j4}, x_{j5}\}$.

Thus, $|S| = 8+4k$ and $\omega(G(9+5k,2) - S) = 6+3k$, and $t(G(9+5k,2)) \leq |S|/\omega(G(9+5k,2) - S) = (8+4k)/(6+3k) = 4/3$.

It is known that $t(G(5,2)) = t(G(9,2)) = 4/3$ (see [1]). Hence, 4/3 is the best upper bound for $t(G(n,2))$ where $n \geq 5$.

Reference

[1] B. Piazza, R. Ringeisen and S. Stueckle, *On the vulnerability of cycle permutation graphs*, ARS Combinatoria **29**(1990), 289–296.