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A Note on Finite Solvable (q)−group and $(s-q)$−group*

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Abstract. In [1], [2], the authors studied the finite solvable group in which every subnormal subgroup is quasinormal (i.e., $(s-q)$−group) and the finite solvable group in which every subnormal subgroup is s−quasinormal (i.e., $(s-q)$−group). In this paper, we shall characterize finite solvable (q)−groups and $(s-q)$−groups by general nilpotent groups and give the classification of inner $(s-q)$−groups.

For convenience, we introduce the following definitions and symbols.

Definition 1 Let H be a subgroup of group G. If $\forall K \leq G, HK = KH$ holds, then H is called quasinormal subgroup of G, written as $H\mathrm{qn}G$, if for any prime $p \mid |G|, P \in \mathrm{syl}p(G), HP = PH$ holds, then H is called an s−quasinormal subgroup of G, written as $H\mathrm{sqn}G$.

$H\mathrm{sn}G$ means that H is a subnormal subgroup of G.

Definition 2 Let G be a finite group. If every subnormal subgroup of G is quasinormal, then G is called (q)−group; if every subnormal subgroup of G is s−quasinormal subgroup of G, then G is called $(s-q)$−group; if every subgroup of G is quasinormal, then G is called quasi-Hamilton group.

Definition 3 Let $\prod(G) = \{p_1, p_2, \cdots, p_n\}$ be a prime factor set of $\mid G \mid$. If there exists Sylow subgroups $P_i \in$ syl $p_i(G)$ $(i = 1, 2, \cdots, n)$ such that $\forall a \in P_i, b \in P_j, <a><b>=<b><a>$ holds, then G is called a general nilpotent group, $\{P_1, P_2, \cdots, P_n\}$ is called a general nilpotent basis.

Definition 4 If every proper subgroup of group G is a $(s-q)$−group, but G is not a $(s-q)$−group, then G is called a inner $(s-q)$−group.

Lemma 1 *Let G be a general nilpotent group and $\{P_1, P_2, \cdots, P_n\}$ a general nilpotent basis of G. Then*

1) *If $Q_i \leq P_i$, then $Q_iP_j = Q_jP_i, i = 1, 2, \cdots, n$;*

2) *If $Q_i \leq P_i$ and $Q_j \leq P_j (i \neq j)$, then $Q_iQ_j = Q_jQ_i$.*

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Proof Assume $a \in Q_j, b \in P_i$, $i \neq j$, then $< a >< b >=< b >< a >$, i.e., $ab = b^t a^s$. Hence $ab \in P_i Q_j$. It follows that $Q_j P_i \subseteq P_i Q_j$. Conversely, it is easy to obtain that $Q_j P_i \subseteq P_i Q_j$, so $P_i Q_j = Q_j P_i$. If $i = j$, obviously, $Q_j P_i = P_i Q_j$. Therefore, 1) holds. 2) follows in a similar way. □

Lemma 2[1] *G is a finite solvable $(q)-$group if and only if G has a Abel normal Hall subgroup N of order odd such that*

1) *G/N is Hamilton group;*

2) $\forall x \in N, \forall y \in G, x^y = x^{k(y)}, k(y)$ *is integer.*

Theorem 1 *G is finite solvable $(q)-$group if and only if G is a general nilpotent group in which every Sylow subgroup is quasi-Hamiltion group.*

Proof Necessity is obvious by Lemma 2.

For the converse part, let $\{P_1, P_2, \cdots, P_n\}$ be a general nilpotent basis of G. Then $G = P_1 P_2 \cdots P_n$. Assume $H\text{sn}G, K \leq G$. Since general nilpotent group is supersolvable group [3, Th. 9.6], we have $K = Q_1 Q_2 \cdots Q_n$, where $Q_i \in \text{syl}p_i(G)$ and if $p_i \nmid |K|, Q_i = 1, 1 \leq i \leq n$. Obviously, it is enough to prove $HQ_i = Q_i H$. By Sylow's Theorem, it can be assumed that $Q_i \subseteq P_i^{x_i}$ (where $x_i \in G), i = 1, 2, \cdots, n$. Thus it is obvious that $\forall x \in G$,

$$H = (H \cap P_1^x)(H \cap P_2^x) \cdots (H \cap P_n^x).$$

Taking in turn $x = x_i$ $(i = 1, 2, \cdots, n)$, we get

$$H = (H \cap P_1^{x_i})(H \cap P_2^{x_i}) \cdots (H \cap P_n^{x_i}).$$

For $H \cap P_j^{x_i} (i \neq j)$, since $H \cap P_j^{x_i} \subseteq P_j^{x_i}$, it follows from Lemma 2 that

$$(H \cap P_j^{x_i}) Q_i = Q_i (H \cap P_j^{x_i}), i = 1, 2, \cdots, n.$$

For $H \cap P_i^{x_i}$, since Q_i and $H \cap P_i^{x_i}$ are all subgroups of $P_i^{x_i}$, $P_i^{x_i}$ is a quasi-Hamilton group, so $(H \cap P_i^{x_i}) Q_i = Q_i (H \cap P_i^{x_i})$. Thus we obtain $Q_i H = HQ_i, i = 1, 2, \cdots, n$, and hence $HK = KH$. So G is a $(q)-$group. □

Lemma 3 *Assume G is a solvable group, $G = NM, N \lhd G$, and $(|M|, |N|) = 1$, if $H\text{sn}G$, then $H = (H \cap M)(H \cap N)$.*

Proof Since $H\text{sn}G$, we may assume $H = H_0 \lhd H_1 \lhd \cdots \lhd H_t \lhd G$. We will prove $H_t = (H_t \cap M)(H_t \cap N)$. Obviously,

$$(H_t \cap M)(H_t \cap N) \subseteq H_t. \tag{1}$$

On the other hand, if $|H_t| \mid |M|$ or $|H_t| \mid |N|$, by Hall's theorem for solvable group, since $H_t \lhd G$, we obtain $H_t \subseteq M$ or $H_t \subseteq N$. In this situation, $H_t = (H_t \cap M)(H_t \cap N)$.

If $|H_t| \not| |M|, |H_t| \not| |N|$, assume $|H_t| = ab$, $a \mid |M|, b \mid |N|$. Since H_t is slovable, so H_t has a Sylow basis. Hence, there exists a subgroup H_1^* of order a and a subgroup H_2^* of order b such that $H_t = H_1^* H_2^*$. Thus there exists some $x \in G$ such that $H_1^* \subseteq M^x, H_2^* \subseteq N$ for any $h \in H_t$, it follow that $h = h_1 h_2$, where $h_1 \in H_1^* \subset M^x, h_2 \in H_2^* \subset N$. Therefore $h = h_1 h_2 \in (H_t \cap M^x)(H_t \cap N)$, i.e., $H_t \subseteq (H_t \cap M^x)(H_t \cap N)$.Since $H_t \lhd G$, so

$$H_t = (H_t \cap M)(H_t \cap N). \tag{2}$$

We obtain from (1) and (2) that $H_t = (H_t \cap M)(H_t \cap N)$.

Since H_t satisfies all the conditions in Lemma 3. It follows by induction that $H = (H \cap M)(H \cap N)$.

Lemma 4[2. Th.2.3] *G is a finite solvable $(s-q)-$group if and if only if G has a normal Abel Hall subgroup N of odd order such that*

1) *G/N is nilpotent group,*

2) *$\forall x \in N, y \in G, x^y = x^{k(y)}, k(y)$ is integer.*

Theorem 2 *G is a finite solvable $(s-q)-$group if and only if G is a general nilpotent group.*

Proof Necessity is trivial by Lemma 4. Conversely, let G be a general nilpotent group, $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where $p_1 < p_2 < \cdots < p_n$ and p_i is prime $(1 \leq i \leq n)$. Thus there exists a general nilpotent basis $\{P_1, P_2, \cdots, P_n\}$ such that $G = P_1 P_2 \cdots P_n$ and $\forall a \in P_i, \forall b \in P_j, <a><b>=<b><a>$.

Let $N = P_2 P_3 \cdots P_n$. Then $N \lhd G, (|P_1|, |N|) = 1$ and $G = P_1 N = P_1^x N, (\forall x \in N)$. Let HsnG, by Lemma 3, $H = (H \cap P_1)(H \cap N)$. Since $H \cap N \lhd H$, so $H \cap N$snG and $H \cap N$snN.

Let $N_1 = P_3 P_4 \cdots P_n$, then $N = P_2 N_1 = P_2^x N_1, (\forall x \in G)$. In the same way, $H \cap N = (H \cap N \cap P_2)(H \cap N \cap N_1) = (H \cap P_2)(H \cap N_1)$.

We proceed in a similar way and finally obtain

$$H = (H \cap P_1)(H \cap P_2) \cdots (H \cap P_n).$$

Similarly,we can obtain that $\forall x \in G$, $H = (H \cap P_1^x)(H \cap P_2^x) \cdots (H \cap P_n^x)$. Take any $p \in \mathrm{syl} P_i(G)$, then there exists some $x \in G$ such that $P = P_i^x$. Since $\{P_1^x, P_2^x, \cdots, P_n^x\}$ is also a general nilpotent basis of G, hence

$$\begin{aligned} PH &= P_i^x H = P_i^x[(H \cap P_1^x)(H \cap P_2^x) \cdots (H \cap P_n^x)] \\ &= (H \cap P_1^x) P_i^x (H \cap P_2^x) \cdots (H \cap P_n^x) = \cdots \\ &= (H \cap P_1^x)(H \cap P_2^x) \cdots (H \cap P_n^x) = HP. \end{aligned}$$

Thus we obtain HsqnG, i.e., G is a solvable $(s-q)-$group. □

Finally, we list the following corollaries without proof.

Corollary 1 *The subgroup, (quotient group) of a general nilpotent group is still a general* *nilpotent group.*

Corollary 2 *G is an inner $(s-q)$-group if and only if G is precisely one of the following:*

I. *q-basic group of order $p^\alpha q^\beta$, the exponent b of $q \bmod p$ is more than 1.*

II. *Inner supersolvable group of order $p^\alpha q^\beta, p^\alpha \mid q-1, \alpha \geq 2$, the definition relation is*

$$a^{p\alpha} = c_1^q = c_2^q = \cdots = c_p^q = 1, c_i c_j = c_j c_i, 1 \leq i, j \leq p.$$

$$c_i^a = c_{i+1}, i = 1, 2, \cdots, p-1, c_p^a = c_1^t, \quad \text{the exponent of } t \bmod q \text{ is } \quad p^{\alpha-1}.$$

III. *The group of order $p^\alpha q^2, p \mid q-1$. The definition relation is*

$$a^{p\alpha} = b_1^q = 1; b_1 b_2 = b_2 b_1,$$

$$b_1^a = b_1^{k_1}, b_2^a = b_2^{k_2}, k_1 \not\equiv k_2 \pmod q), k_1^p \equiv k_2^p \pmod q).$$

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有限可解(q)群与$(s-q)$群的一个注记

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摘　要

文[1]. [2]分别研究了每个次正规子群为拟正规的有限群（即(q)群）以及每个次正规子群为$s-q$拟正规的有限群（即$(s-q)$群）. 本文利用广幂零群的概念对(q)群与$(s-q)$群给出了一个新的刻划，并得到内$(s-q)$群的完全分类.