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On Derivation Approximation of Bernstein Operator*

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If f is a function defined on $[0,1]$, then the Bernstein polynomial $B_n(f;x)$ of f is as follows

$$B_n(f;x) := \sum_{k=0}^{n} f(\frac{k}{n})P_{nk}(x), P_{nk}(x) := \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}. \tag{1}$$

If f is bounded on $[0,1]$, x is a point if discontinuty of the frist kind, Herzog and Hill[1] proved

$$\lim_{n\to\infty} B_n(f;x) = \frac{1}{2}(f(x^+)+f(x^-)). \tag{2}$$

Furthermore, Cheng[2] gave the rate of convergence above when f is bounded variation on $[0,1]$. A question we are interested in is that if $f' \in BV[0,1]$, what is B_n' approximate to f'? What is error estimation we have obtained?

Throrem *Let $f' \in BV[0,1]$. Then for $x \in (0,1)$,*

$$\begin{aligned} & \mid B_n'(f;x) - \frac{1}{2}(f'(x^+)+f'(x^-)) \mid \\ \leq & \frac{10\bigvee_0^1(f') + 15\|f'\|}{2[(n-1)x(1-x)]^{\frac{1}{2}}} + \frac{(x(1-x))^{-1}}{n-1}\sum_{k=1}^{n-1}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{(1-x)}{\sqrt{k}}}(g_x') \\ + & \frac{13\mid f'(x^+)-f'(x^-)\mid}{4[(n-1)x(1-x)]^{\frac{1}{2}}}, \end{aligned}$$

where $\|f'\| = \max_{0\leq x\leq 1}\mid f'(x)\mid$, $\bigvee_a^b(g_x')$ is total variation of g_x' on $[a,b]$, $g_x'(t)$ as follows

$$g_x'(t) = \begin{cases} f'(t)-f'(x^+), & x < t \leq 1, \\ 0, & t = x, \\ f'(t)-f'(x^-), & 0 \leq t < x. \end{cases} \tag{3}$$

Proof Let $x \in (0,1)$ be a only point of discontinuity of the frist kind. Then there exist $k_0, 0 < k_0 < n$, such that $x \in (\frac{k_0}{n}, \frac{k_0+1}{n})$. Using

$$B_n'(f;x) = n\sum_{k=0}^{n-1}(f(\frac{k+1}{n}) - f(\frac{k}{n}))P_{n-1,k}(x),$$

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we get

$$
\begin{aligned}
& \mid B_n'(f;x) - \frac{1}{2}(f'(x^+) + f'(x^-)) \mid \\
\leq\ & \mid \{\sum_{k=0}^{k_0-1} + \sum_{k=K_0+1}^{n-1} + \sum_{k=k_0}\}[(n(f(\frac{k+1}{n}) - f(\frac{k}{n})) - f'(\frac{k}{n-1})]P_{n-1,k}(x)] \mid \\
+\ & \mid \sum_{k=0}^{n-1} f'(\frac{k}{n-1})P_{n-1,k}(x) - \frac{1}{2}(f'(x^+) + f'(x^-)) \mid \\
=\ & A + B = \mid A_1 + A_2 + A_3 \mid + B.
\end{aligned}
$$

By $\frac{k}{n} < \frac{k}{n-1} \leq \frac{k+1}{n}, k = 0, 1, ..., n-1$, and

$$
P_{n,k'}(x) \leq \frac{5}{2}(nx(1-x))^{\frac{-1}{2}}, \tag{4}
$$

we have

$$
\mid A_1 \mid \leq \frac{5}{2[(n-1)x(1-x)]^{\frac{1}{2}}} \sum_{k=0}^{k_0-1} \mid f'(\xi_k) - f'(\frac{k}{n-1}) \mid \leq \frac{5\bigvee_0^1(f')}{2[(n-1)x(1-x)]^{\frac{1}{2}}},
$$

where $\frac{k}{n} < \xi_k < \frac{k+1}{n}, k \leq k_0 - 1$. Similarly, we have

$$
\mid A_2 \mid \leq \frac{5\bigvee_0^1(f')}{2[(n-1)x(1-x)]^{\frac{1}{2}}}.
$$

On using (4) we get

$$
\begin{aligned}
\mid A_3 \mid \ & \leq \ [\mid n(f(\frac{k_0+1}{n}) - f(x)) + n(f(x) - f(\frac{k_0}{n})) \mid + \mid f'(\frac{k_0}{n-1}) \mid] P_{n-1,k_0}(x) \\
& \leq \ \frac{15\|f'\|}{2[(n-1)x(1-x)]^{\frac{1}{2}}}.
\end{aligned}
$$

If x is the end of $[\frac{k_0}{n}, \frac{k_0+1}{n}]$, then

$$
\mid A \mid = \mid \sum_{k=0}^{n-1}(f'(\xi_k) - f'(\frac{k}{n-1}))P_{n-1,k}(x) \mid \leq \frac{5\bigvee_0^1(f')}{2[(n-1)x(1-x)]^{\frac{1}{2}}}.
$$

On using

$$
f'(t) = \frac{1}{2}(f'(x^+) + f'(x^-)) + g_x'(t) + \frac{1}{2}(f'(x^+) - f'(x^-))\mathrm{sgn}(t-x),
$$

we have

$$
\begin{aligned}
B \ = \ & \mid B_{n-1}(f';x) - \frac{1}{2}(f'(x^+) + f'(x^-)) \mid \leq \mid B_{n-1}(g_x';x) \mid \\
& + \frac{1}{2} \mid f'(x^+) - f'(x^-) \mid \mid B_{n-1}(\mathrm{sgn}(t-x);x) \mid .
\end{aligned}
$$

From [1,2], we have

$$| B_{n-1}(g_x'; x) | \leq \frac{3(x(1-x))^{-1}}{n-1} \sum_{k=1}^{n-1} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{1-x}{\sqrt{k}}} (g_x'),$$

and

$$| B_{n-1}(\mathrm{sgn}(t-x); x) | = | \{ \sum_{\frac{k}{n-1}>x} - \sum_{\frac{k}{n-1}<x} \} P_{n-1,k}(x) | \leq \frac{4}{[(n-1)x(1-x)]^{\frac{1}{2}}}.$$

If $x = \frac{k'}{n}$, then

$$\begin{aligned} | B_{n-1}(\mathrm{sgn}(t-x); x) | &\leq | \{ \sum_{\frac{k}{n-1}>x} - \sum_{\frac{k}{n-1}<x} \} P_{n-1,k}(x) | + | P_{n-1,k'}(x) | \\ &\leq \frac{13}{2[(n-1)x(1-x)]^{\frac{1}{2}}}. \end{aligned}$$

Therefore,we have

$$B \leq \frac{3(x(1-x))^{-1}}{n-1} \sum_{k=1}^{n-1} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{(1-x)}{\sqrt{k}}} (g_x') + \frac{13}{[4(n-1)x(1-x)]^{\frac{1}{2}}} | f'(x^+) - f'(x^-) | .$$

The theorem is proved. □

Finally, we point out that the results we have obtained in this paper can not be improved.

References

[1] F.Herzog and J.D.Hill,Amer.J.Math, 68(1946), 109-124.

[2] Cheng FuHua, J.Approxi. Th.39:3(1983),259-274.

[3] G.G.Lorete, Bernstein polynomials, Toronto, 1953.