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The Inverse Generalized Eigenvalue Problem*

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Abstract. This paper presents a kind of inverse generalized eigenvalue problem for real symmetric band matrix, and gives a proof to the existence of the solution to the problem for Jacobi and ordinary symmetric matrices.

1. Introduction

In this paper, we consider a kind of inverse generalized eigenvalue problem as follows:

Problem IGE: Given a real symmetric band matrix $B = (b_{ij})_{i,j=1}^{n}$, which is positive definite and $b_{ij} = 0$ when $| i - j |> r$. Given real numbers $\left\{\lambda_i^{(k)}\right\}_{i=1}^{k}$, $k = n - r, \cdots, n$, satisfying

$$\lambda_i^{(k)} \leq \lambda_i^{(k-1)} \leq \lambda_{i+1}^{(k)}, \quad i = 1, 2, \cdots, k-1;\ k = n - r, \cdots, n. \tag{1}$$

Determine a real symmetric matrix $A = (a_{ij})_{i,j=1}^{n}$, so that the generalized eigenvalue problem

$$A(k)x = \lambda B(k)x$$

has eigenvalues $\left\{\lambda_i^{(k)}\right\}_{i=1}^{k}$, where $A(k) = (a_{ij})_{i,j=1}^{k}$, $B(k) = (b_{ij})_{i,j=1}^{k}$, $a_{ij} = 0$ when $| i - j |> r$.

This problem often arises in molecular spectroscopy, where the inverse of matrix B is called the Wilson Kinematic matrix determined by molecular geometry and atomic masses, and matrix A the force constant matrix, to be determined by the measured spectrum. In structural mechanics, A and B are called mass matrix and stiffness matrix respectively, so the problem is how to determine the mass distribution of a structure by its stiffness distribution and its natural frequencies of vibration under boundary conditions. (Because of the reciprocity of matrix A and B in generalized eigenvalue problem in certain sense, this problem can also be interpreted as how to determine the stiffness distribution of a structure,by its mass distribution and its natural frequencies of vibration). In the following two sections, we are going to discuss two special cases:$r = n - 1$ and $r = 1$. For $r = n - 1$, which means that A and B are real symmetrix matrices, we first give a result about the inverse eigenvalue problem of real symmetric matrix obtained by Friedland and then transform the inverse generalized eigenvalue problem into inverse eigenvalue

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problem, thereby to obtain the results of the existence and the number of the solutions of the problem. For $r=1$, namely, A and B are Jacobi matrices, we assume that $\{\lambda_i^{(k)}\}_{i=1}^{k}$ satisfies

$$\lambda_i^{(k)} < \lambda_i^{(k-1)} < \lambda_{i+1}^{(k)}, \quad i=1,\cdots,k-1; k=n-r,\cdots,n \tag{2}$$

and give the proof of existence of the solution of the problem, this can be regarded as a generalization of the result of Hald [1].

2. The inverse generalized eigenvalue problem for real symmetric matrices

If $r=n-1$, the problem IGE can be stated as follows:

Let $B=(b_{ij})_{i,j=1}^{n}$ be a real valued and positive definite matrix. Given sequence of real numbers $\{\lambda_i^{(k)}\}_{i=1}^{k}$, $k=1,\cdots,n$ satisfying the inequalities (1). Reconstruct a real symmetric matrix A so that the generalized eigenvalue problem

$$A(k)x = \lambda B(k)x \tag{3}$$

has solution $\{\lambda_i^{(k)}\}_{i=1}^{k}$, where $A(k)=(a_{ij})_{i,j=1}^{k}$, $B(k)=(b_{ij})_{i,j=1}^{k}$, $k=1,\cdots,n$.

If B is a unit matrix, the problem becomes an inverse eigenvalue problem to which Friedland [2] proved the following theorem:

Theorem 1 *Let $\{\lambda_i^{(k)}\}_{i=1}^{k}$, $k=1,\cdots,n$ be sequences of real numbers satisfying (1). Then there exists an $n\times n$ real valued symmetric matrix A such that $\{\lambda_i^{(k)}\}_{i=1}^{k}$ are the eigenvalues of matrix $A(k)=(a_{ij})_{i,j=1}^{k}$,* for $k=1,\cdots,n$.

Let $P_k(\lambda)=\prod_{i=1}^{k}(\lambda-\lambda_i^{(k)})$. Then, the number of matrices A is finite if and only if whenever α is a root of $P_k(\lambda)$ of multiplicity $m>1$ then α is a root of $P_{k+1}(\lambda)$ of multiplicity not less than m, for $k=2,\cdots,n-1$. Assume that the given eigenvalue problem has a finite number of solutions. Let l_k be the number of simple roots of $P_k(\lambda)$ which are not roots of $P_{k+1}(\lambda)$ for $k=1,\cdots,n-1$. Then the number of distinct matrices which satisfy the condition of the theorem is equal to 2^l, where

$$l=\sum_{k=1}^{n-1} l_k.$$

In particular, if there hold strict inequalities in (1), then the number of distinct matrices A is equal to $2^{n(n-1)/2}$.

Now we return to the case where B is a real valued and positive definite matrix. Let $B=LL^T$ be the Cholesky decomposition of B. It is known from Theorem 1 that under condition (1), there exists a real symmetric matrix A' such that $\{\lambda_i^{(k)}\}_{i=1}^{k}$ are the eigenvalues of matrix $A'(k)$, for $k=1,\cdots,n$.

Let

$$A=LA'L^T. \tag{4}$$

Partition matrix A' and L as

$$A' = \begin{bmatrix} A'(k) & x_k^T \\ X_k & Y_k \end{bmatrix}, \quad L = \begin{bmatrix} L(k) & 0 \\ M_k & N_k \end{bmatrix},$$

where $A'(k)$ and $L(k)$ are kk matrices. Because

$$A = \begin{bmatrix} L(k) & 0 \\ M_k & N_k \end{bmatrix} \begin{bmatrix} A'(k) & X_k^T \\ X_k & Y_k \end{bmatrix} \begin{bmatrix} L(k)^T & M_k^T \\ 0 & N_k^T \end{bmatrix} = \begin{bmatrix} L(k)A'(k)L(k)^T & * \\ * & * \end{bmatrix},$$

$$B = \begin{bmatrix} L(k) & 0 \\ M_k & N_k \end{bmatrix} \begin{bmatrix} L(k)^T & M_k^T \\ 0 & N_k^T \end{bmatrix} = \begin{bmatrix} L(k)L(k)^T & * \\ * & * \end{bmatrix},$$

where $*$ denotes a suitable submatrix, we have

$$A(k) = L(k)A'(k)L(k)^T \tag{5}$$

$$B(k) = L(k)L(k)^T. \tag{6}$$

Hence, the generalized eigenvalue problem (3) becomes

$$L(k)A'(k)L(k)^T x = \lambda L(k)L(k)^T x, \tag{7}$$

since B is positive definite $L(k)$ is reversible, let $L(k)^T x = y$, (7) becomes

$$A'(k)y = \lambda y. \tag{8}$$

Thus (3) does has solution $\{\lambda_i^{(k)}\}_{i=1}^k$, $k = 1, \cdots, n$. This means that $A = LA'L^T$ is the matrix which we want to reconstruct. Moreover, since A and A' have one to one relationship, the number of distinct solution A is the number of distinct solution A'.

The above discussion can be stated as a theorem about the inverse generalized eigenvalue problem for real symmetric matrix:

Theorem 2 *Let $\{\lambda_i^{(k)}\}_{i=1}^k$, $k = 1, \cdots, n$ be sequences of real numbers satisfying the inequalities (1). Let $B = (b_{ij})_{i,j=1}^n$ be a real symmetric and positive definite matrix. Then there exists an $n \times n$ real valued symmetric matrix $A = (a_{ij})_{i,j=1}^n$, such that $\{\lambda_i^{(k)}\}_{i=1}^k$ are the solutions of generalized eigenvalue problem (3) for $k = 1, \cdots, n$. Let $B = LL^T$ be the Cholesky decomposition of matrix B and matrix A' such that $A'(k)$ has eiigenvalues $\{\lambda_i^{(k)}\}_{i=1}^k$ for $k = 1, \cdots, n$. Then matrix $A = LA'L^T$ is a solution of the problem, the necessary and sufficient condition of the number of solutions beging finite and the number of solutions when it is finite are the same as that in Theorem 1.*

3. The inverse generalized eigenvalue problem for Jacobi matrix

When $r = 1$, the problem IGE is called the inverse generalized eigenvalue problem for Jacobi matrix, about which we have the following theorem:

Theorem 3 *Let B be a positive definite Jacobi matrix. Given real numbers $\{\alpha_i\}_{i=1}^{n}$, $\{\beta_i\}_{i=1}^{n-1}$ satisfying*

$$\alpha_1 < \beta_1 < \cdots < \alpha_{n-1} < \beta_{n-1} < \alpha_n. \tag{9}$$

Then there exists a Jacobi matrix A such that the generalized eigenvalue problem

$$A(k)x = \lambda B(k)x, \ \ k = n-1, n$$

has solution $\{\alpha_i\}_{i=1}^{n}$, $\{\beta_i\}_{i=1}^{n-1}$, where $A(k) = (a_{ij})_{i,j=1}^{k}$, $B(k) = (b_{ij})_{i,j=1}^{k}$.

We need the following lemmas in the proof of Theorem 3.

Lemma 1 *Let*

$$A = \begin{bmatrix} a_1 & b_1 & & \\ b_1 & \ddots & \ddots & \\ \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{bmatrix}, \quad B = \begin{bmatrix} c_1 & d_1 & & \\ d_1 & \ddots & \ddots & \\ \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & d_{n-1} \\ & & d_{n-1} & c_n \end{bmatrix}$$

and B be positive definite. Then

$$f_k(\lambda) = \det(\lambda B(k) - A(k)) = t_k P_k(\lambda), \tag{10}$$

where $P_k(\lambda)$ is a monic polynomial of degree $k, t_k = detB(k) > 0$, and t_k is given by the recurrence formula

$$\begin{cases} t_k = c_k t_{k-1} - d_{k-1}^2 t_{k-2}, & k = 2, \cdots, n, \\ t_0 = 1, \ \ t_1 = c_1. \end{cases} \tag{11}$$

Proof From $\det B(k) \det(\lambda B(k) - A(k)) = \det(\lambda I - B(k)^{-1}A(k))$, we have (10). Expanding $\det B(k)$ by the last row gives (11). □

Lemma 2 *Let $\{\alpha_i\}_{i=1}^{n-1}, \{\beta_i\}_{i=1}^{n-1}$ satisfy (9). Let*

$$P_n(\lambda) = \prod_{i=1}^{n}(\lambda - \alpha_i), \ \ P_{n-1}(\lambda) = \prod_{i=1}^{n-1}(\lambda - \beta_i). \tag{12}$$

Then the equation

$$P_n'(\lambda)P_{n-1}(\lambda) - P_n(\lambda)P_{n-1}'(\lambda) = c[P_{n-1}(\lambda)]^2 \tag{13}$$

has at least one real root for any real constant $c > 1$. Where $P_n'(\lambda)$ denotes the derivative of $P_n(\lambda)$.

Proof It is obvious that β_i $(i = 1, \cdots, n-1)$ is not the solution of (13). Hence the equation is equivalent to

$$\varphi(\lambda) \equiv \frac{P_n'(\lambda)P_{n-1}(\lambda) - P_n(\lambda)P_{n-1}'(\lambda)}{[P_{n-1}(\lambda)]^2} = c.$$

From (12), we have

$$\frac{P(\lambda)}{P_{n-1}(\lambda)} = \lambda + c_0 + \sum_{i=1}^{n-1} \frac{c_i}{\lambda - \beta_i}.$$

Thus

$$\varphi(\lambda) = \left[\frac{P_n(\lambda)}{P_{n-1}(\lambda)}\right]' = 1 - \sum_{i=1}^{n-1} \frac{c_i}{(\lambda - \beta_i)^2}, \tag{14}$$

where $c_i = \frac{P_n(\beta_i)}{P'_{n-1}(\beta_i)}$, especially

$$c_{n-1} = \frac{P_n(\beta_{n-1})}{P'_{n-1}(\beta_{n-1})} = \frac{\prod_{i=1}^{n}(\beta_{n-1} - \alpha_i)}{\prod_{i=1}^{n-2}(\beta_{n-1} - \beta_i)} < 0.$$

Therefore, we have $\lim_{\lambda \to \beta_{n-1}^{+0}} \varphi(\lambda) = +\infty$, it follows that there exists a constant $l > \beta_{n-1}$ such that $\varphi(l) > c$. It is obvious from (14) that $\lim_{\lambda \to +\infty} \varphi(\lambda) = 1$, this gives that there exists a constant $u > l$, such that $\varphi(u) < c$. Since $\varphi(\lambda)$ is continuous on $[l, u]$, there exists a point λ_0 such that $\varphi(\lambda_0) = c$. The proof is complete.

Lemma 3 *Let*

$$B = \begin{bmatrix} c_1 & d_1 & & & \\ d_1 & \ddots & \ddots & & \\ \ddots & \ddots & \ddots & \ddots & \\ \ddots & \ddots & \ddots & d_{n-1} & \\ & & d_{n-1} & c_n & \end{bmatrix}$$

be positive definite. Let $P_n(\lambda)$ and $P_{n-1}(\lambda)$ satisfy the condition in Lemma 2, and t_k $(k = 1, \cdots, n)$ be defined by (11). Then there exist real numbers a and b such that

$$t_n P_n(\lambda) = (c_n\lambda - a)t_{n-1}P_{n-1}(\lambda) - (d_{n-1}\lambda - b)^2 t_{n-2}P_{n-2}(\lambda), \tag{15}$$

where $P_{n-2}(\lambda)$ is a monic polynomial of degree $n-2$, which has $n-2$ real roots interlaced by $n-1$ real roots of $P_{n-1}(\lambda)$.

Proof First we assume $d_{n-1} \neq 0$. Let $\phi(\lambda) = t_n P_n(\lambda) - (c_n\lambda - a)t_{n-1}P_{n-1}(\lambda)$. Now we determine the constant a so that $\phi(\lambda)$ has a real root of multiplicity two. To do this, we only need to solve the equations about a and λ:

$$\phi(\lambda) = t_n P_n(\lambda) - (c_n\lambda - a)t_{n-1}P_{n-1}(\lambda) = 0, \tag{16}$$

$$\phi'(\lambda) = t_n P'_n(\lambda) - c_n t_{n-1}P_{n-1}(\lambda) - (c_n\lambda - a)t_{n-1}P'_{n-1}(\lambda) = 0. \tag{17}$$

Multiplying (16) by $P'_{n-1}(\lambda)$, and multiplying (17) by $P_{n-1}(\lambda)$ and substracting, we obtain $t_n P'_n(\lambda)P_{n-1}(\lambda) - t_n P_n(\lambda)P'_{n-1}(\lambda) - c_n t_{n-1}[P_{n-1}(\lambda)]^2 = 0$, i.e.,

$$P'_n(\lambda)P_{n-1}(\lambda) - P_n(\lambda)P'_{n-1}(\lambda) = \frac{c_n t_{n-1}}{t_n}[P_{n-1}(\lambda)]^2. \tag{18}$$

From (11)

$$t_n = c_n t_{n-1} - d_{n-1}^2 t_{n-2} \tag{19}$$

and $d_{n-1} \neq 0$ by assumption, $t_{n-2} > 0$ $t_n > 0$, according to Lemma 1, we have

$$\frac{c_n t_{n-1}}{t_n} = 1 + \frac{d_{n-2}^2 t_{n-2}}{t_n} > 1.$$

From Lemma 2 it follows that (18) has a real root $\lambda_0 > \beta_{n-1}$. Substituting $\lambda = \lambda_0$ in (16) gives

$$a = c_n\lambda_0 - \frac{t_n P_n(\lambda_0)}{t_{n-1}P_{n-1}(\lambda_0)}.$$

Suppose $b = d_{n-1}\lambda_0$, then

$$\phi(\lambda) = (\lambda - \lambda_0)^2 g(\lambda) = -(d_{n-1}\lambda - b)^2 t_{n-2}P_{n-2}(\lambda),$$

where $g(\lambda)$ is a polynomial of λ and so is $P_{n-2}(\lambda)$.

Comparing the leading coefficients of both sides in the above equation, noticing the definition of $\phi(\lambda)$ and (19), it follows that $P_{n-2}(\lambda)$ is a monic polynomial of $n-2$ and that (15) is valid.

Substituting $\lambda = \beta_j$ in (15) gives

$$t_n \prod_{i=1}^{n}(\beta_j - \alpha_i) = 0 - (d_{n-1}\beta_j - b)^2 t_{n-2}P_{n-2}(\beta_j) = -d_{n-1}^2(\beta_j - \lambda_0)^2 t_{n-2}P_{n-2}(\beta_j),$$

By (9), $\operatorname{sgn} P_{n-2}(\beta_j) = (-1)^{n-j+1}$, therefore $P_{n-1}(\lambda)$ has a root v_j in the interval $(\beta_j,\ \beta_{j+1})$ for $j = 1, \cdots, n-1$. i.e.,

$$\beta_1 < v_1 < \beta_2 < \cdots < \beta_{n-2} < v_{n-2} < \beta_{n-1}.$$

When $d_{n-1} = 0$, (15) becomes

$$t_n P_n(\lambda) = (c_n\lambda - a)t_{n-1}P_{n-1}(\lambda) - b^2 t_{n-2}P_{n-2}(\lambda) \tag{20}$$

and (19) becomes

$$t_n = c_n t_{n-1} \tag{21}$$

so (20) can be rewritten as

$$P_n(\lambda) = (\lambda - a/c_n)P_{n-1}(\lambda) - b^2(t_{n-2})/t_n P_{n-2}(\lambda). \tag{22}$$

The proof is complete except that of (22) which we refer to Lemma 1 in [1] of Hald. □

Proof of Theorem 3 For $n = 2$, by Lemma 1, we have to find real numbers a_1, b_1, a_2 satisfying

$$\begin{aligned} f_1(\lambda) &= \lambda c_1 - a_1 = c_1(\lambda - \beta_1), && (23)\\ f_2(\lambda) &= (\lambda c_2 - a_2)(\lambda c_1 - a_1) - (\lambda d_1 - b_1)^2 = (c_1c_2 - d_1^2)(\lambda - \alpha_1)(\lambda - \alpha_2) && (24) \end{aligned}$$

Comparing the coefficients in both sides of (23) gives

$$c_1a_2 + c_2a_1 - 2d_1b_1 = (c_1c_2 - d_1^2)(\alpha_1 + \alpha_2) \tag{25}$$

$$a_1a_2 - b_1^2 = (c_1c_2 - d_1^2)\alpha_1\alpha_2. \tag{26}$$

From (23), we have $a_1 = c_1\beta_1$. Substituting in (25) and (26) and eliminating a_2 gives $-c_1c_2\beta_1^2 + 2d_1\beta_1b_1 - b_1^2 = (c_1c_2 - d_1^2)[\alpha_1\alpha_2 - \beta_1(\alpha_1+\alpha_2)]$. i.e.,

$$b_1^2 - 2d_1\beta_1b_1 + c_1c_2\beta_1^2 + (c_1c_2 - d_1^2)[\alpha_1\alpha_2 - \beta_1(\alpha_1+\alpha_2)] = 0.$$

This is a quadratic equation of b_1 with discriminant

$$\begin{aligned}\Delta &= 4d_1^2\beta_1^2 - 4c_1c_2\beta_1^2 - 4(c_1c_2 - d_1^2)[\alpha_1\alpha_2 - \beta_1(\alpha_1+\alpha_2)] \\ &= 4(c_1c_2 - d_1^2)[-\beta_1^2 + (\alpha_1+\alpha_2)\beta_1 - \alpha_1\alpha_2] \\ &= 4(c_1c_2 - d_1^2)(\beta_1 - \alpha_1)(\alpha_2 - \beta_1).\end{aligned}$$

Because of the positive definiteness of B and the interlacing condition (9) , we have $\Delta > 0$. So the quadratic equation has two distinct roots. Substituting one of them in (25), we can determine a_2, and witha_1, b_1, a_2 we can construct matrix A.

We assume that the theorem is true for $n \le m-1$. Because

$$\alpha_1 < \beta_1 < \cdots < \alpha_{m-1} < \beta_{m-1} < \alpha_m,$$

$$P_m(\lambda) = \prod_{i=1}^{m}(\lambda - \alpha_i),\ P_{m-1}(\lambda) = \prod_{i=1}^{m-1}(\lambda - \beta_i).$$

By Lemma 3, there exist real numbers a_m and b_{m-1} such that

$$t_mP_m(\lambda) = (c_m\lambda - a_m)t_{m-1}P_{m-1}(\lambda) - (d_{m-1}\lambda - b_{m-1})^2t_{m-2}P_{m-2}(\lambda).$$

Where t_k $(k = m, m-1, m-2)$ are defined by (11), $P_{m-2}(\lambda)$ is a monic polynomial of degree $n-2$, whose roots $\{v_i\}_{i=1}^{m-2}$ are real and interleave $\{\beta_i\}_{i=1}^{m-1}$.An induction assumption guarantees a Jacobi matrix of order $m-1$,

$$\tilde{A} = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & \ddots & \ddots & & \\ \ddots & \ddots & \ddots & \ddots & \\ \ddots & \ddots & \ddots & b_{m-2} \\ & & b_{m-2} & a_{m-1} \end{bmatrix}$$

such that

$$\det(\lambda B(m-1) - \tilde{A}) = t_{m-1}P_{m-1}(\lambda); \det(\lambda B(m-2) - \tilde{A}(m-2)) = t_{m-2}P_{m-2}(\lambda)$$

Now, construct matrix A as

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & \ddots & \ddots & & \\ \ddots & \ddots & \ddots & \ddots & \\ \ddots & \ddots & \ddots & b_{m-2} & \\ \ddots & \ddots & b_{m-2} & a_{m-1} & b_{m-1} \\ & & & b_{m-1} & a_m \end{bmatrix}.$$

It is bovious that

$$\det(\lambda B - A) = (c_m\lambda - a_m)t_{m-1}P_{m-1}(\lambda) - (d_{m-1}\lambda - b_{m-1})^2 t_{m-2}P_{m-2}(\lambda) = t_m P_m(\lambda),$$

$$\det(\lambda B(m-1) - A(m-1)) = t_{m-1}P_{m-1}(\lambda).$$

Therēfore matrix A is what we want, and this finally completes the proof of Theorem 3. □

References

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一类广义特征值反问题

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摘　要

本文提出了一个实对称带状矩阵的广义特征值反问题，并且证明了对于 Jacobi 矩阵和一般对称矩阵，问题的存在性.