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Some Properties of Dynamical Systems on the Torus*

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Abstract This is a talk on seminar given respectively at the University of London, the University of Warwick and the University of Floyence during February to May of 1990. The author would like to thank Professors I.C. Percival, D.K. Arrowsmith, S. Bullett, W. Parry, A. Manning and R. Conti for their encouragement and help.

In this talk I am going to give a brief introduction to some aspects of my recent work. This concerns some very common questions in dynamical systems on the torus. In the book "Nonlinear Differential Equations" by G. Sansone and R. Conti (1964), topic of differential equations on the torus were referred to the following three papers:

[1] A.N. Kolmogorov, *Dynamical systems with an integral invariant on the torus*, Dokl. Acad. Nauk SSSR, **93**(1953), 763–766.

[2] S. Sternberg, *On differential equations on the torus*, Amer. J. Math.,**79**(1957), 397–402.

[3] E.J. Akutowicz, *The ergodic property of the characteristics on a torus*, Quart. J. Math., **(2) 9**(1958), 275–281.

My work is related to these papers. I shall split the talk into three sections. The first section is the global structure of trajectories for differential equations on the torus with an integral invariant; the second section is the ergodicity of these systems with respect to the invariant measure given by the integral invariant, and finally, the third section is the unique ergodicity of dynamical systems on the torus.

Section 1 The global structure of trajectories for differential equations on the torus with an integral invariant.

Consider a system of ordinary differential equations

$$\frac{dx}{dt} = X(x), \quad x \in S(\text{phase space}). \tag{1}$$

Suppose $f(p,t)$ is the flow defined by (1).

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Definition 1 If there is a function $M(x) > 0$ such that $\int_D M(x)dx = \int_{D_t} M(x)dx$ for any domain $D \subset S$ and $D_t = f(D,t), \forall t \in R^1$, then (1) is called a system which possesses an integral invariant .

Consider the differential system defined on a torus T^2

$$\frac{dx}{dt} = X(x,y), \qquad \frac{dy}{dt} = Y(x,y). \tag{2}$$

Suppose that this system satisfies the following conditions:

H1. $X, Y \in C^1$, and $X(x+1,y) = X(x,y+1) = X(x,y), Y(x+1,y) = Y(x,y+1) = Y(x,y)$.

H2. There exists an integral invariant $U(x,y) \in C^1$ and $U(x+1,y) = U(x,y+1) = U(x,y)$.

H3. The singular points are isolated.

In the case $X^2 + Y^2 > 0$, it is shown in ([1], X, Y are analytic) and in ([2], $X, Y \in C^n$) that the global topological structure of trajectories for (2) is the same as for the system

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = \alpha. \quad \text{(where } \alpha \text{ is a real number)}$$

In the general case, let

$$\rho = \frac{\int_0^1 \int_0^1 U(x,y)Y(x,y)dxdy}{\int_0^1 \int_0^1 U(x,y)X(x,y)dxdy}, \tag{3}$$

then we have the following theorems (see [4]: Yu Shuxiang, J. of Differential Equations, **53:2**(1984), 277–287).

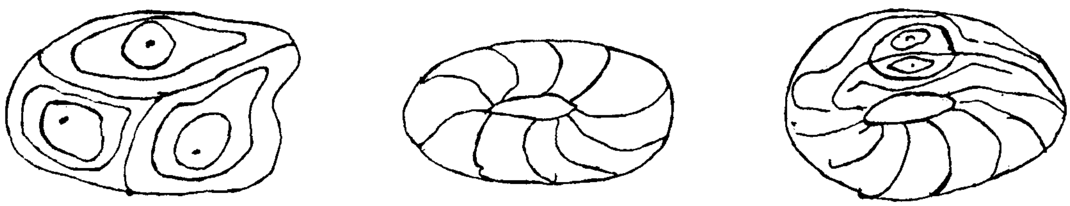
Theorem 1 *Suppose that the system (2) satisfies H1, H2 and H3. Then all orbits, except singular points and separatrices connecting them, are closed curves if and only if ρ is a rational number.*

Theorem 2 *Suppose that the system (2) satisfies H1, H2 and H3. If ρ is an irrational number and (2) has no center type of singular points, then there are orbits which are dense in the torus.*

Therefore, we see that the qualitative behavior of the flow is classified in terms of the rationality of ρ and the analysis for singular points. For any given differential system there is a simple criterion to determine to which type it belongs.

Qualitative behaviour

(a) the rational structure type
(b) the completely dense type
(c) the mixed type (a mixture of dense domains and rational domains)



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Now let us turn to the second question.

Section 2 The ergodicity of the system (2).

Suppose that $f: \ T^2 \times R^1 \longrightarrow T^2$ is a flow defined by (2). By the definition of an integral invariant, we know that

$$\mu(A) = \frac{\int\int_A U(x,y)dxdy}{\int\int_{T^2} U(x,y)dxdy}, \quad \forall A \subset T^2 \tag{4}$$

is an invariant measure for the flow f.

A problem is: Whether or not the completely dense type is ergodic with respect to $\mu(A)$?

To this question, an answer is the following (see [5]: Chen Yong-Hong, Acta Mathematica Sinica,**30:1**(1987), 111–114)

Theorem 3 *Suppose that the system (2) satisfies H1, H2 and H3. Then f is ergodic with respect to $\mu(A)$ if and only if ρ is an irrational number and (2) has no center type of singular points.*

In [5], the author also proved the following result for the flow which does not necessarily possess an integral invariant.

Theorem 4 *Suppose that $g: \ T^2 \times R^1 \longrightarrow T^2$ is a C^1–flow defined on the torus T^2. If it satisfies the conditions (2) the singular points are isolated, and (3) there is a nontrivial P stable trajectory γ such that the closure $\overline{\gamma} = T^2$.*

Then the dynamical system (T^2, g) has at most one nontrivial ergodic invariant measure.

In the case of a torus, Theorem 4 is a generalization of a result of Katok (see [6]: A.B. Katok, Dokl. Acad. Nauks SSR, **211(4)**(1973), 775–778). Because Theorem 4 does not require the condition that there are no singular points.

Now let me move on to the third question.

Section 3 The unique ergodicity of dynamical systems on the torus.

If φ is a measure preserving transformation of a probability space S with measure μ, then Birkhoff's Ergodic Theorem concluded that for every function $f \in L^1$ the following limit (time average) exists almost everywhere with respect to μ:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\varphi^k x) = \bar{f}(x), \quad \text{a.e.} \tag{5}$$

However, in practice, for a given point $x_0 \in S$, this theorem can not tell us whether the time average $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\varphi^k x_0)$ exists, even when φ is ergodic with respect to μ. So, it is of significance to find some conditions under which the time average exists everywhere. This question is closely related to that of the unique ergodicity of a transformation.

Suppose that S is a compact metric space and (S,φ) is a (topological) dynamical system. Suppose $V(\varphi)$ is the set of all normalized Borel measure invariant with respect ot φ. A well known result of Kryloff and Bogoliouboff is that $V(\varphi)$ is non-empty.

Definition 2 A dynamical system (S,φ) is said to be uniquely ergodic if it has precisely one normalized Borel invariant measure. Similarly a flow (S,ϕ_t) is uniquely ergodic if it has precisely one normalized Borel invariant measure.

Now we suppose that S is a torus T^2. E.J. Akutowicz in [3] proved that every C^2-flow which has neither singular points nor closed orbits is uniquely ergodic.

This result has not been improved for a long time. It was still cited in the book "An Introduction to the Theory of Smooth Dynamical Systems" by Wieslaw Szlenk (1984).

Problem: Is the C^2-smooth condition necessary?

First, Han Mao-an (see [7], Acta Mathematica Sinica, New Series, **4:4**(1988), 338–342) proved that if φ_t is a C^1-flow or a C^0-flow satisfying certain additional condition on the torus, then φ_t is uniquely ergodic whenever it has neither singular points nor closed orbits.

Recently, I obtained the following result (see [8], Yu Shuxiang, Chinese Ann. of Math., **10A:5**(1989), 214–216)

Theorem 5 *Suppose φ_t is a continuous flow on the torus T^2. If there are no rest points and closed orbits, then φ_t is uniquely ergodic.*

环面上动力系统的某些性质

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摘 要

本文讨论了曲面动力系统中三个方面的主题，即，环面上具有积分不变量的动力系统轨线的拓扑分类，这类系统的各态历经性质，以及环面上动力系统的唯一各态历经性. 文章简要地叙述了在这些主题上最近得到的新结果.