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Fixed Point Theorems for Set-Valued Mappings*

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1 Introduction

Set-valued mappings, as we know, are of great use in the theory of operations research, especially in the existence of optimal solutions, stability theory and convergence of algorithms for mathematical programming, and game theory, etc.. In this paper, we give a result for set-valued mappings defined on a paracompact convex set which is closely related to Theorem 8 in [1], and neither of them implies the other. From this we give some sufficient conditions for the existence of an equilibrium point, and new coincidence or fixed point theorems, which generalize the earlier results in the references.

Let E be a Hausdorff topological vector space and X be a non-empty convex subset of E, E', the dual space of E. For $x \in E$, let us put

$$\begin{aligned} P_X(x) &= \{\phi \in E' \mid \phi(x) \leq \inf_{y \in X} \phi(y)\}, \\ N_X(x) &= \{\phi \in E' \mid \phi(x) \geq \inf_{y \in X} \phi(y)\}. \end{aligned}$$

Then we have that $P_X(x) = -N_X(x)$. $P_X(x)$ and $N_X(x)$ are called the positive normal and the negative normal cone to X at x respectively. We denote by $T_X(x)$ the negative polar cone to $N_X(x)$, then $T_X(x) = \mathrm{cl}[\cup_{\lambda>0}\lambda(X - x)]$, and $T_X(x)$ is a common tangent cone to X at x while X is $\mathrm{cl}(X)$.

Let Y be a Hausdorff vector space, $F : X \to 2^Y$ be a set-valued mapping. We recall that F is said to be upper hemicontinuous (u.h.c.) on X, if for every $\phi \in Y'$, the function

$$\sigma(F(x), \phi) = \sup_{y \in F(x)} \phi(y)$$

is upper semicontinuous on X. F is called weakly upper hemicontinuous (w.u.h.c.) on X, if for every $\phi \in Y'$, the set $\{x \in X \mid \sigma(F(x), \phi) < 0\}$ is open in X. It is easy to see that u.h.c. $\Rightarrow$ w.u.h.c., but the converse is not true. F is called upper demicontinuous (u.d.c.) on X, if for every $x_0 \in X$ and any open half- space $H = \{y \in Y \mid \phi(y) < r\}$ in Y containing $F(x_0)$, where $0 \neq \phi \in Y'$ and r is a real number, there exists a neighborhood $V(x_0)$ of x_0 in X such that $F(x) \subset H$ for all $x \in V(x_0)$. It is not difficult to prove that

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u.d.c. $\Rightarrow$ u.h.c.. The following examples will show that the converse is false. We take $E = X = Y = E^2$, define F as

$$F(x) = \{y \in Y \mid y_2 \geq \exp(-y_1) - \| x \|\}, x \in X.$$

For any $\phi \in Y'$, let ϕ be in the form $\phi(y) = a_1 y_1 + a_2 y_2$. Then we have $\sigma(F(x), \phi) = \sigma(F(0), \phi) - a_2 \| x \|$, for all $x \in X$, and F is u.h.c. on X. The open half-space $\{y \in Y \mid y_2 > 0\}$ containing $F(0)$ does not contain $F(x)$, for all $x \neq 0$.

2 General Results

Lemma 1 *Let X be a non-empty set in a topological vector space, and g a real-valued function on $X \times X$ such that*

(a) *For each fixed $x \in X$, the set $\{y \in X \mid g(x, y) > 0\}$ is compactly open in X.*

(b) *For each fiexed $x \in X$, the set $\{y \in X \mid g(x, y) > 0\}$ is convex.*

(c) *$g(x, x) \leq 0$ for all $x \in X$.*

d) *For some compact convex set $X_0 \subset X$, the set $\{y \in X \mid g(x, y) \leq 0$ for all $x \in X_0\}$ is compact.*

Then there exists $y_1 \in X$ such that $g(x, y_1) \leq 0$ for all $x \in X$.

Proof This is a special case of Theorem 1.1' in [8].

Lemma 2[9] *Let E, Y be locally convex Hausdorff topological vector spaces, and $\{p\}, \{q\}$ the systems of semi-norms respectively defining the topologies of E and Y. Then a linear operator T on $D(T) \subset E$ into Y is continuous if and only if, for every semi- norm $q \in \{q\}$, there exists a semi-norm $p \in \{p\}$ and a positive number β such that*

$$q(Tx) \leq \beta p(x) \text{ for all } x \in D(T).$$

Lemma 3 *Let E, Y be locally convex Hausdorff topological vector spaces, $\mathcal{L}_s(E, Y)$ the set of all continuous operators from E to Y with simple convergence topology, and $X \subset E$ a non-empty subsets of E. If $L : X \to \mathcal{L}_s(E, Y)$ is a continuous mapping associating with each $x \in X$ an element of $\mathcal{L}_s(E, Y)$, then the mapping $L' : X \to \mathcal{L}_s(Y', E')$, where $L'(x)$ is the dual operator of $L(x)$, is also continuous.*

Proof Let p be any semi-norm in $\mathcal{L}_s(Y', E')$ and $x, x_0 \in X$. Then $p(L'(x) - L'(x_0)) = \sup_{1 \leq i \leq n} q[(L'(x) - L'(x_0))y_i']$, where q is a semi-norm in E'_{ω^*} and $y_i' \in Y'$ for each i, and also

$$\begin{aligned} q[(L'(x) - L'(x_0))y_i'] &= \sup_{1 \leq j \leq m} |(L'(x) - L'(x_0))y_i'(x_j)| \\ &= \sup_{1 \leq j \leq m} |y_i'((L(x) - L(x_0))x_j)|, \end{aligned}$$

where $x_j \in E$ for each j. Apply Lemma 2 to each y_i', then there exists a semi-norm t_i in Y and a constant β_i for each i such that

$$|y_i'((L(x)-L(x_0))x_j)| \le \beta_i t_i[(L(x)-L(x_0)x_j)].$$

Therefore

$$\begin{aligned} p[L'(x)-L'(x_0)] &\le \sup_{1\le i\le n}\sup_{1\le j\le m} \beta_i t_i[(L(x)-L(x_0))x_j] \\ &= \sup_{1\le i\le n}\beta_i \sup_{1\le j\le m} t_i[(L(x)-L(x_0))x_j] \\ &= \sup_{1\le i\le n}\beta_i r_i[L(x)-L(x_0)], \end{aligned}$$

where r_i is a semi-norm in $\mathcal{L}_s(E,Y)$ with

$$r_i(L) = \sup_{1\le j\le m} t_i(L(x_j)), \text{ for } L\in\mathcal{L}_s(E,Y).$$

So we have $p(L'(x)-L'(x_0)) \to 0$ as $x\to x_0$. Since L is continuous in x and the proof is complete.

Lemma 4[10] *Let X be a non-empty set in a locally convex Hausdorff topological vector space and $p: X\to E'$ a continuous mapping. For each $x\in X$, define $h(x,\cdot): X\to R$ by $h(x,y)=p(y)(y-x)$ for every $y\in X$, then for each fixed $x\in X$ the function $h(x,\cdot)$ is continuous on every compact subset of X.*

Theorem 1 *Let E, Y be locally convex Hausdorff topological vector spaces and X be a non-empty paracompact convex subset of E, and $L: X\to\mathcal{L}_s(E,Y)$ be a continuous mapping as in Lemma 2. Let Φ be a non-empty convex subset of Y' and $S: X\to 2^{\Phi}$ be a set-valued mapping such that*

(a) *For each $x\in X, S(x)$ is a non-empty convex subset of Φ.*

(b) *For each $\phi\in\Phi$, the set $S^{-1}(\phi)=\{x\in X \mid \phi\in S(x)\}$ is open in X.*

Then either (1) $\{y\in X \mid S(y)\cap L'(y)^{-1}P_X(y)\ne\emptyset\}\ne\emptyset$, or (2) for any compact convex subset X_0 of X and any compact subset K of X, $\{y\in X \mid S(y)\cap L'(y)^{-1}P_{X_0}(y)\ne\emptyset\}\not\subset K$.

Proof By (a), for each $z\in X$, there is a $\phi_z\in\Phi$ such that $\phi_z\in S(z)$, so $z\in S^{-1}(\phi_z)$. By (b), $\{S^{-1}(\phi_z) \mid z\in X\}$ is an open covering of X. Let $\{V_i \mid i\in I\}$ be a locally finite refinement of this open covering and $\{\alpha_i \mid i\in I\}$ be a continuous partition of unity, subordinated to $\{V_i \mid i\in I\}$, i.e., for each $i\in I, \alpha_i: X\to[0,1]$ is continuous and $\mathrm{cl}(\alpha_i^{-1}(0,1])\subset V_i$, each $x\in X$ has a neighborhood meeting only a finite number of $\mathrm{cl}(\alpha_i^{-1}(0,1])$, and $\sum_{i\in I}\alpha_i(x)=1$ for every $x\in X$. Since $\{V_i \mid i\in I\}$ is a refinement of $\{S^{-1}(\phi_z) \mid z\in X\}$, there exists for each $i\in I$ a $z_i\in X$ such that $V_i\subset S^{-1}(\phi_{z_i})$. Now define

$$\psi(x)=\sum_{i\in I}\alpha_i(x)\phi_{z_i} \quad (x\in X).$$

Then we have $\psi(x) \in \Phi$ for each $x \in X$, and for $i \in I$ and $x \in X$, $\alpha_i(x) > 0$ implies $x \in \alpha_i^{-1}(0,1] \subset V_i \subset S^{-1}(\phi_{z_i})$, i.e., $\phi_{z_i} \in S(x)$. Therefore by the convexity of $S(x)$ we have

$$\psi(x) \in S(x) \ \text{ for all } x \in X. \tag{1}$$

Define $p : X \to E'$ by

$$p(y) = \sum_{i \in I} \alpha_i(y) L'(y) \phi_{z_i} \ (y \in X)$$

and $h : X \times X \to R$ by

$$h(x,y) = \psi(y)[L(y)(y-x)], (x,y) \in X \times X.$$

Then by Lemma 3 and Lemma 4, we have that, for each fixed $x \in X, h(x,y) = p(x)(y-x)$ is continuous in y in any compact subset of X, so that for every $x \in X$, $\{y \in X \mid h(x,y) > 0\}$ is compactly open in X. It is clear that $\{x \in X \mid h(x,y) > 0\}$ is convex for every $y \in X$ and $h(x,x) = 0$ for all $x \in X$. By Lemma 1, we have either

(i) There exists $y_1 \in X$ such that $h(x,y_1) \le 0$ for all $x \in X$.

or

(ii) For any non-empty compact convex subset X_0 of X and any compact subset K of X, $\{y \in X \mid h(x,y) \le 0$ for all $x \in X_0\} \not\subset K$.

For any subset C of X, since

$$h(x,y) = \psi(y)[L(y)(y-x)] = [L'(y)\psi(y)]y - [L'(y)\psi(y)]x,$$

it follows that

$$\begin{aligned} & h(x,y) \le 0 \text{ for all } x \in C \Longleftrightarrow \mathrm{L}'(y)\psi(y) \in P_C(y) \\ & \Longleftrightarrow \psi(y) \in L'(y)^{-1}P_C(y) \Longrightarrow S(y) \cap L'(y)^{-1}P_C(y) \ne \emptyset \text{ (by (1))}. \end{aligned}$$

So we have the requirement from (i) and (ii).

Remark In Theorem 1, we can replace $P_X(y)$ and $P_{X_0}(y)$ by $N_X(y)$ and $N_{X_0}(y)$ respectively, the theorem is also true. The same remark applies to all the following theorems and corollaries.

Theorem 2 *Let E and Y be Hausdorff topological vector spaces and X be a paracompact convex subset of E, and K be a non-empety compact convex subset of X. Let $L : X \to \mathcal{L}_s(E,Y)$ be a continuous mapping, and $H : X \to 2^Y$ be a w.u.h.c. set valued mapping such that for each $x \in X$, $H(x)$ is a non-empty subset of Y. Then at least one of the following conditions holds:*

(a) *There exists $x_0 \in X$ such taht $\{0\}$ and $H(x_0)$ can not be strictly separated by a closed hyperplane in E.*

(b) *There exist $x_1 \in K$ and $\phi_1 \in L'(x_1)^{-1}P_X(x_1)$ such that $\sigma(H(x_1),\phi_1) < 0$.*

(c) *There exist $x_2 \in X \setminus K$ and $\phi_2 \in L'(x_2)^{-1}P_K(x_2)$ such that $\sigma(H(x_2),\phi_2) < 0$.*

Proof Suppose that (a) does not occur. Then for every $x \in X$, there is $\phi_x \in Y'$ such that $\sigma(H(x), \phi_x) < 0 = \phi_x(0)$. We take $\Phi = Y'$ and define the mapping $S : X \to 2^{\Phi}$ as follows:

$$S(x) = \{\phi \in \Phi \mid \sigma(H(x), \phi) < 0\}, x \in X.$$

By weakly upper hemicontinuity of H, the set $S^{-1}(\phi)$ is open in X for every $\phi \in \Phi$. From Theorem 1, either (i) there exist $x_1 \in X$ and $\phi_1 \in S(x_1)$ such that $\phi_1 \in L'(x_1)^{-1}P_X(x_1)$, or (ii) there exist $x_2 \in X \setminus K$ and $\phi_2 \in S(x_2)$ with $\phi_2 \in L'(x_2)^{-1}P_K(x_2)$. For (i), (b) is valid if $x_1 \in K$, and (c) is valid if $x_1 \in X \setminus K$, since $\phi_1 \in L'(x_1)^{-1}P_X(x_1) \subset L'(x_1)P_K(x_1)$ and $\phi_1 \in S(x_1)$. For (ii), condition (c) is valid. This completes the proof.

Remark If $L'(x)$ is inveritble for every $x \in \text{int}(X)$, then Theorem 2 remains true if condition (b) is replaced by (b'): There exist $x_1 \in K \cap \text{Bd}X$ and $\phi_1 \in L'(x_1)^{-1}P_X(x_1)$ such that $\sigma(H(x_1), \phi_1) < 0$. The same remark applies to the following theorems and corollaries.

Corollary *Let E, Y, X, K, L be the same as in Theorem 2. Let $F : X \to 2^Y$ and $G : X \to 2^Y$ be u.h.c. set-valued mappings on X such that $F(x) \neq \emptyset, G(x) \neq \emptyset$, for every $x \in X$. Then at least one of the following conditions holds:*

(a) *There exists $x_0 \in X$ such that $F(x_0)$ and $G(x_0)$ can not be strictly separated by a closed hyperplane in E.*

(b) *There exist $x_1 \in K$ and $\phi_1 \in L'(x_1)^{-1}P_X(x_1)$ such that $\sigma(F(x_1), \phi_1) + \sigma(G(x_1), -\phi_1) < 0$.*

(c) *There exist $x_2 \in X \backslash K$ and $\phi_2 \in L'(x_2)^{-1}P_K(x_2)$ such that $\sigma(F(x_2), \phi_2) + \sigma(G(x_2), -\phi_2) < 0$.*

Proof Take $H = F - G$. Since F and G are u.h.c., H is u.h.c. on X. For every $x \in X$, $\{0\}$ and $H(x)$ can be strictly separated iff that $F(x)$ and $G(x)$ can be strictly separated. Hence the corollary follows from Theorem 2.

3 Existence of Equilibrium Point and Coincidence Theorems

Theorem 3 *Let E be a Hausdorff topological vector space and X be a paracompact vonvex subset of E, and K be a non-empty compact convex subset of X. Let Y be a locally convex Hausdorff topological vector space and $L : X \to \mathcal{L}_s(E, Y)$ be a continuous mapping. Let $H : X \to 2^Y$ be a w.u.h.c. set-valued mapping such that the following conditions are satisified*

(a) *For each $x \in X$, $H(x)$ is a non-empty closed convex subset of Y.*

(b) *For any $x \in K$ and $\phi \in L'(x)^{-1}P_X(x)$, $\sigma(H(x), \phi) \geq 0$.*

(c) *For any $x \in X \setminus K$ and $\phi \in L'(x)^{-1}P_K(x)$, $\sigma(H(x), \phi) \geq 0$.*

Then there exists a point $x_0 \in X$ such that $0 \in H(x_0)$.

Proof By Theorem 2, there exists a point $x_0 \in X$ such that $\{0\}$ and $H(x_0)$ can not be strictly separated. Since Y is locally convex and $\{0\}$ is compact and $H(x_0)$ is closed convex, we have $0 \in H(x_0)$. This completes the proof.

Theorem 3 generalizes a theorem of Simons [7, Theorem 3.1, p.1138].

Lemma 5 *Let E, Y, X, L be as in Theorem 3 and C a convex subset of X. Let $H : X \to 2^Y$ be a set-valued mapping such that for each $x \in X$, $H(x)$ is a non-empty convex subset of Y. Then*

$$\sum_{\substack{x\in X\\ \phi\in Y'}} d_\phi[H(x), cl(L(x)T_C(x))] = 0 \Longleftrightarrow \sigma(H(x),\phi) \geq 0 \text{ for all } \phi \in L'(x)^{-1}P_C(x) \text{ and all } x \in X,$$

where $d_\phi(A, B) = \inf\{|\phi(x-y)| \mid x \in A \text{ and } y \in B\}$ as in [10].

Proof $\Longrightarrow$. If it is not true, then there exist $x \in X$ and $\phi_0 \in L'(x)^{-1}P_C(x)$ such that $\sigma(H(X), \phi_0) < 0$, i.e., for each $y \in H(x), \phi_0(y) < 0$.

On the other hand, $-L'(x)\phi_0 \in T_C^-(x)$ since $\phi_0 \in L'(x)^{-1}P_C(x)$. So that $\phi_0(L(x)z) = (L'(x)\phi_0)z \geq 0$ for all $z \in T_C(x)$, i.e., $\phi_0(u) \geq 0$ for all $u \in L(x)T_C(x)$. Then we have

$$\begin{aligned} &d_{\phi_0}[H(x), \mathrm{cl}(L(x)T_C(x))] = d_{\phi_0}[H(x), L(x)T_C(x)] \\ &= \inf_{u\in L(x)T_C(x)} \phi_0(u) - \sup_{y\in H(x)} \phi_0(y) \geq -\sigma(H(x),\phi_0) > 0, \end{aligned}$$

which is a contradiction.

$\Longleftarrow$. For $x \in X$ and $\phi \in Y'$, if $\inf_{u\in H(x)} |\phi(u)| = 0$, then $d_\phi(H(x), \mathrm{cl}(L(x)T_C(x))) \leq \inf_{u\in H(x)} |\phi(u)| = 0$ since $0 \in \mathrm{cl}(L(x)T_C(x))$. So we can suppose $\inf_{u\in H(x)} |\phi(u)| > 0$, it follows that $0 \notin H(x)$ and either $\sup_{y\in H(x)} \phi(y) < 0$ or $\inf_{y\in H(x)} \phi(y) > 0$, since if there exist $y_1, y_2 \in H(x)$ with $\phi(y_1) < 0$ and $\phi(y_2) > 0$. Then take $\lambda = \phi(y_2)/[\phi(y_2) - \phi(y_1)]$ and $y_0 = \lambda y_1 + (1-\lambda)y_2$, we have $\lambda \in (0,1)$ and $y_0 \in H(x)$ since $H(x)$ is convex and that $\phi(y_0) = 0$ gives a contradiction.

If $\sup_{y\in H(x)} \phi(y) < 0$, i.e., $\sigma(H(x),\phi) < 0$, then $\phi \notin L'(x)^{-1}P_C(x), L'(x)\phi \notin P_C(x) = -T_C^-(x)$, so there exists $u \in T_C(x)$ with $\phi(L(x)u) = (L'(x)\phi)u < 0$. Let $y = L(x)u$, then $y \in L(x)T_C(x)$ and $\phi(y) < 0$. For every $v \in H(x)$, there exists $\lambda > 0$ such that $\phi(v) = \lambda\phi(y) = \phi(\lambda y)$ since $\phi(v) < 0$. Therefore $\lambda y = L(x)(\lambda u) \in L(x)T_C(x)$ implies $d_\phi[H(x), \mathrm{cl}(L(x)T_C(x))] \leq |\phi(v - \lambda y)| = 0$.

For the case $\inf_{y\in H(x)} \phi(y) < 0$, i.e., $\sigma(H(x), -\phi) < 0$, we can repeat the above argument by replacing ϕ by $-\phi$.

Corollary 1 *Let E, X, K, Y, L, H be as in Theorem 3, and the following conditions be satisfied:*

(a) *For each $x \in X, H(x)$ is a non-empty closed convex subset of Y.*

(b) $\sum_{\substack{x\in X\\ \phi\in Y'}} d_\phi[H(x), cl(L(x)T_X(x))] = 0.$

(c) $\sum_{\substack{x\in X\setminus K\\ \phi\in Y'}} d_\phi[H(x), cl(L(x)T_X(x))] = 0.$

Then there exists $x_0 \in X$ *such that* $0 \in H(x_0)$.

Corollary 1 is a generalization of a theorem in [2, Theorem 6, p.232].

Corollary 2 *Let* E, X, K, Y, L *be as in Theorem 3. Let* $F : X \to 2^Y$ *and* $G : X \to 2^Y$ *be u.h.c. set-valued mappings such that*

(a) *For each* $x \in X$, $F(x)$ *and* $G(x)$ *are non-empty closed convex subsets of* Y, *at least one of which is compact.*

(b) $\sum_{\substack{x\in K\\ \phi\in Y'}} d_\phi[F(x) - G(x), cl(L(x)T_X(x))] = 0.$

(c) $\sum_{\substack{x\in X\setminus K\\ \phi\in Y'}} d_\phi[F(x) - G(x), cl(L(x)T_K(x))] = 0.$

Then there exists $x_0 \in X$ *such that* $F(x_0) \cap G(x_0) \neq \emptyset$.

Proof Take $H = F - G$. The corollary follows from Corollary 1.

Corollary 3 *Let* E, X, K, Y, L, F, G *be as in Corollary 2, and let the following conditions be satisfied:*

(a) *For each* $x \in X, F(x)$ *and* $G(x)$ *are non-empty closed convex subsets of* Y, *at least one of which is compact.*

(b) *For any* $x \in K$ *and* $\phi \in L'(x)^{-1}P_X(x), \sigma(F(x), \phi) + \sigma(G(x), -\phi) \geq 0$.

(c) *For any* $x \in X \setminus K$ *and* $\phi \in L'(x)^{-1}P_K(x), \sigma(F(x), \phi) + \sigma(G(x), -\phi) \geq 0$.

Then there exists $x_0 \in X$ *with* $F(x_0) \cap G(x_0) \neq \emptyset$.

Proof It is an immediate consequence of Theorem 3 with $H = F - G$.

From the remark after Theorem 2, Corollary 2 and Coroallary 3 are generalizations of Theorem 10 and Theorem 9 in [1] respectively. Corollary 3 is also a generalization of the main result in [3], as well as Theorem 3.4 in [10] since, for any non-empty subsets K, L of X, (K, L, F, G) is admissible if and only if for every $x \in K$ and every $\phi \in N_L(x), \sigma(F(x), \phi) + \sigma(G(x), -\phi) \geq 0$.

Theorem 4 *Let* E, X, K, Y, L, H *be as in Theorem 3, and the following conditions be satisfied:*

(a) *For each* $x \in X$, $H(x)$ *is a non-empty closed convex subset of* Y.

(b) *For any* $x \in K$ *and* $\phi \in L'(x)^{-1}P_X(x)$, *one has*

$$\inf_{u\in L^{-1}(x)H(x)} \inf_{h>0} \inf_{z\in X} |L'(x)\phi(x + hu - z)|/h = 0.$$

(c) *For any* $x \in X \setminus K$ *and* $\phi \in L'(x)^{-1}P_K(x)$, *one has*

$$\inf_{u\in L^{-1}(x)H(x)} \inf_{h>0} \inf_{z\in K} |L'(x)\phi(x + hu - z)|/h = 0.$$

Then there exists $x_0 \in X$ *such that* $0 \in H(x_0)$.

Proof It suffices to verify the conditions (b) and (c) of Theorem 3. For a non-empty convex subset C of X and any $x \in X$ and $\phi \in L'(x)^{-1}P_C(x)$, we have $L'(x)\phi \in P_C(x)$, i.e., $L'(x)\phi(x-z) \le 0$, for all $z \in C$. Suppose $\sigma(H(x),\phi) < 0$. It follows that $\phi(L(x)u) < 0$, for all $u \in L^{-1}(x)H(x)$. Hence $|L'(x)\phi(x+hu-z)| = L'(x)\phi(z-x)-h\phi(L(x)u) \ge -h\phi(L(x)u)$, for all $z \in C$ and $h > 0$ and $u \in L^{-1}(x)H(x)$. Therefore we have

$$\inf_{u\in L^{-1}(x)H(x)} \inf_{h>0} \inf_{z\in C} |L'(x)\phi(x+hu-z)| \ge -\sigma(H(x),\phi) > 0.$$

This contradiction proves the theorem.

Corollary *Let* E, X, K, Y, L, F, G *be as in Corollary 2 of Theorem 3, and the following conditions be satisfied:*

(a) *For each* $x \in X, F(x)$ *and* $G(x)$ *are non-empty closed convex subsets of* Y, *at least one of which is compact.*

(b) *For any* $x \in K$ *and* $\phi \in L'(x)^{-1}P_X(x)$,

$$\inf_{\substack{u\in L^{-1}(x)F(x)\\ v\in L^{-1}(x)G(x)}} \inf_{h>0} \inf_{z\in X} |L'(x)\phi(x+h(u-v)-z)|/h = 0.$$

(c) *For any* $x \in X \setminus K$ *and* $\phi \in L'(x)^{-1}P_K(x)$,

$$\inf_{\substack{u\in L^{-1}(x)F(x)\\ v\in L^{-1}(x)G(x)}} \inf_{h>0} \inf_{z\in K} |L'(x)\phi(x+h(u-v)-z)|/h = 0.$$

Then there exists $x_0 \in X$ *with* $F(x_0) \cap G(x_0) \neq \emptyset$.

The above corollary generalizes the main result in [4].

4 Fixed Point Theorems

We shall take $Y = E$ throughout this section.

Theorem 5 *Let* E *be a locally convex Hausdorff topological vector space and* X *be a paracompact convex subset of* E *and* K *be a non- empty compact convex subset of* X. *Let* $L : X \to \mathcal{L}_s(E,E)$ *be a continuous mapping and* $F : X \to 2^E$ *be an u.h.c. set-valued mapping such that the following conditions are satisfied:*

(a) *For each* $x \in X, F(x)$ *is a non-empty closed convex subset of* E.

(b) *For any* $x \in K$ *and* $\phi \in L'(x)^{-1}P_X(x), \sigma(F(x),\phi) \ge \phi(x)$ *(or* $\sigma(F(x),-\phi)+\phi(x) \ge 0$*).*

(c) *For any* $x \in X\setminus K$ *and* $\phi \in L'(x)^{-1}P_K(x), \sigma(F(x),\phi) \ge \phi(x)$ *(or* $\sigma(F(x),-\phi)+\phi(x) \ge 0$*).*

Then there exists $x_0 \in X$ such that $x_0 \in F(x_0)$.

Proof Take $H(x) = F(x) - x$ (or $H(x) = x - F(x)$) for $x \in X$, then the theorem follows from Theorem 3.

Theorem 5 generalizes a result in [3, Theorem 2.2 & Theorem 2.4].

Theorem 6 *Let E, X, K, L, F be as in Theorem 5 and suppose that*

(a) *For each $x \in X, F(x)$ is a non-empty closed convex subset of E.*

(b)

$$\sum_{\substack{x \in K \\ \phi \in Y'}} d_\phi(F(x), cl[L(x)T_X(x) + x]) = 0,$$

or

$$\sum_{\substack{x \in X \setminus K \\ \phi \in Y'}} d_\phi(F(x), cl[x - L(x)T_X(x)]) = 0,$$

(c)

$$\sum_{\substack{x \in K \\ \phi \in Y'}} d_\phi(F(x), cl[L(x)T_K(x) + x]) = 0,$$

or

$$\sum_{\substack{x \in X \setminus K \\ \phi \in Y'}} d_\phi(F(x), cl[x - L(x)T_K(x)]) = 0.$$

Then there exists $x_0 \in X$ with $x_0 \in F(x_0)$.

Proof Take $H(x) = F(x) - x$ (or $H(x) = x - F(x)$) for $x \in X$, then the theorem follows from Corollary 1 of Theorem 3.

For $x \in E$, define

$$L_X(x) = \{y \in E \mid y = x + \alpha(z - x) \text{ for some } z \in X \text{ and } \alpha \geq 0\},$$
$$O_X(x) = \{y \in E \mid y = x - \alpha(z - x) \text{ for some } z \in X \text{ and } \alpha \geq 0\},$$

Then cl $[I_X(x)] = x + T_X(x)$ and cl $[O_X(x)] = X - T_X(x)$

Corollary *Let E, X, K, F be as in Theorem 5. Suppose that*

(a) *For each $x \in X, F(x)$ is a non-empty closed convex subset of E.*

(b)

$$\sum_{\substack{x \in K \cap Bdx \\ \phi \in Y'}} d_\phi(F(x), cl[I_X(x))]) = 0$$

or

$$\sum_{\substack{x\in K\cap BdX\\ \phi\in Y'}} d_\phi(F(x), cl[O_X(x)]) = 0$$

(c)

$$\sum_{\substack{x\in X\setminus K\\ \phi\in Y'}} d_\phi(F(x), cl[I_X(x)]) = 0$$

or

$$\sum_{\substack{x\in X\setminus K\\ \phi\in Y'}} d_\phi(F(x), cl[O_X(x)]) = 0$$

Then there exists $x_0 \in X$ with $x_0 \in F(x_0)$.

Proof Note that remark after Theorem 2, this is a special case of Theorem 6 when L is taken as an identity operator on E.

This Corollary is Theorem 3.5 in [10] which generalizes the results in [5] and [6], as well as fixed point theorems for inward and outward mappings.

Theorem 7 *Let E, X, K, F be as in Theorem 5. Suppose*

(a) *For each $x \in X$, $F(x)$ is a non-empty closed convex subset of E.*

(b) *For any $x \in K \cap BdX$ and $\phi \in P_X(x)$,*

$$\inf_{u\in F(x)} \inf_{h>0} \inf_{z\in X} |\phi(x + h(u - x) - z)|/h = 0.$$

(c) *For any $x \in X \setminus K$ and $\phi \in P_K(x)$,*

$$\inf_{u\in F(x)} \inf_{h>0} \inf_{z\in X} |\phi(x + h(u - x) - z)|/h = 0.$$

Then there exists $x_0 \in X$ such that $x_0 \in F(x_0)$.

Proof This is a special case of Corollary of Theorem 4, if we take $L(x)$ as an identity operator and $G(x) = \{x\}$ for all $x \in X$.

Theorem 7 generalizes a result in [4] which is a generalization of Reich's fixed point theorem [5].

5 Matching Theorems

Theorem 8 *Let X be a paracompact convex subset in a Hausdorff topological vector space E and K be a non-empty compact convex subset of X. Let Y be a locally convex Hausforff topological vector space and $L : X \to \mathcal{L}_s(E, Y)$ be a continuous mapping. Let*

$\{A_i \mid i \in I\}$ and $\{B_j \mid j \in J\}$ be two locally finite families of relatively closed subsets of X such that $\cup_{i \in I} A_i = \cup_{j \in J} B_j = X$. Let $\{C_i \mid i \in I\}$ and $\{D_j \mid j \in J\}$ be two corresponding families of non-empty subsets of Y such that, for each $x \in X$, either $\cup_{x \in A_i} C_i$ or $\cup_{x \in B_j} D_j$ is contained in a compact convex subset of Y. Suppose that for each $x \in X$, there exists $(i, j) \in I \times J$ such that $x \in A_i \cap B_j$ and

$$\sum_{\substack{x \in K \\ \phi \in Y'}} d_\phi(C_i - D_j, cl[L(x)T_X(x)]) = 0,$$

$$\sum_{\substack{x \in X \setminus K \\ \phi \in Y'}} d_\phi(C_i - D_j, cl[L(x)T_K(x)]) = 0.$$

Then there exist two non-empty finite subsets $I_0 \subset I$ and $J_0 \subset J$ such that $(\cap_{i \in I_0} A_i) \cap (\cap_{j \in J_0} B_j) \neq \emptyset$ and

$$clco(\cup_{i \in I_0} C_i) \cap clco(\cup_{j \in J_0} D_j) \neq \emptyset,$$

where $clco(A)$ denotes the closed convex hull of a set A.

Using Corollary 2 of Theorem 3, the proof is analoguous to the proof of Theorem 11 in [1]. In the same way, we have the following result.

Theorem 9 *Let E, X, K, Y, L be as in Theorem 8 and $\{A_i \mid i \in I\}$ be a locally finite family of relatively closed subsets of X with $\cup_{i \in I} A_i = X$. Let $\{C_i \mid i \in I\}$ be a corresponding family of non-empty subsets of Y. Suppose that for each $x \in X$, there exists $i \in I$ such that $x \in A_i$ an*

$$\sum_{\substack{x \in K \\ \phi \in Y'}} d_\phi(C_i, cl[L(x)T_X(x)]) = 0,$$

$$\sum_{\substack{x \in X \setminus K \\ \phi \in Y'}} d_\phi(C_i, cl[L(x)T_K(x)]) = 0.$$

Then there exists a non-empty finite subset $I_0 \subset I$ such that $\cap_{i \in I_0} A_i \neq \emptyset$ and $0 \in clco(\cup_{i \in I_0} C_i)$.

Theorem 8 and Theorem 9 are generalizations of the corresponding results in [1].

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集值映象的不动点定理

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摘　要

本文讨论了定义在仿紧凸集上的集值映象的有关性质,并由此得到了集值映象的平衡点和不动点存在性的若干充分条件,推广了一些已有的结果.