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Normalizing Extensions and Modules*

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Introduction

Throughout this paper all rings are associative and have an identity, and all modules are right unitary modules unless otherwise indicated. Also, the phrase R is a subring of S will always imply that R and S have the same identity. If A is a ring and N_A is a submodule of the A-module M_A the notation $N_A|M_A$ means that N_A is a direct summand of M_A.

Suppose that R is a subring of the ring S. The ring S is a normalizing extension of R if there is a finite set $\{a_1, a_2, \cdots, a_n\} \subseteq S$ such that $S = \sum_{i=1}^n Ra_i$ with $Ra_i = a_iR$ for each i, and S is a free normalizing extension of R if in addition $a_1 = 1$ and S is free with basis $\{a_1, a_2, \cdots, a_n\}$ as both a right and left R-module. S is an excellent extension of R, if S is a free normalizing extension of R and S is R-projective:that is, if N_S is a submodule of M_S, then $N_R|M_R$ implies $N_S|M_S$.

There are several papers to discuss the relationship between R-modules and S-modules when S is a normalizing extension or an excellent extension of R, for instance, see [1,2,4,5]. In this paper, we will continue these investigations.

When a ring S is an extension of a ring R with the same identity, for every R-module M, $M\otimes_R S$ and $\mathrm{Hom}_R(S, M)$ are S-modules under the natural module operations. If σ is an automorphism of the ring R and M is an R-module, we can define another R-module structure by the law "$\circ$" $m \circ r = mr^\sigma$ for $m \in M, r \in R$ and denote this R-module as M^σ.

Let S be a free normalizing extension of R with basis $\{a_1 = 1, a_2, \cdots, a_n\}$. Then for each a_j, there is an automorphism σ_i of $R : \sigma(r) = r'$, where $a_ir = r'a_i, r, r' \in R$. Therefore,$M \otimes_R S = \oplus_{i=1}^n (M \otimes a_i) \cong M^{\sigma_i}$ and $\mathrm{Hom}_R(S, M) = \oplus_{i=1}^n \mathrm{Hom}_R(a_iR, M) \cong \oplus_{i=1}^n M^{\sigma_i^{-1}}$, as R-modules.

Semisimple modules

By [1, Theorem 3], we knew that if S is an excellent extension of R, then R-module M semisimple implies S-modules $M \otimes_R S$ semisimple and S-module M_S semisimple implies R-module M_R semisimple. Now, we show that the converse is true.

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Proposition 1 *Let S be an excellent extension of R.*

(1) If M is an $R-$module, then $R-$module $M\otimes_R S$ semisimple implies $R-$module M semisimple.

(2) If M is an $S-$module, then $R-$module M_R semisimple implies $S-$module M_S semisimple.

Proof (1). If $(M\otimes_R S)_S$ is semisimple, then $R-$module $(M\otimes_R S)_R$ is semisimple. So, $M_R \cong M\otimes 1 \subseteq (M\otimes_R S)_R$ is semisimple.

(2). Let $S-$module N_S be a submodule of M_S. Then N_R as an $R-$module is a submodule of M_R. Since M_R is semisimple, $N_R|M_R$. Therefore, $N_S|M_S$ and this proves that M_S is semisimple. □

Similarly, we also have

Proposition 2 *If S is an excellent extension of R and M is an $R-$module, then $S-$module $Hom_R(S,M)$ is semisimple if and only if $R-$module M is semisimple.*

Proof By [1, Theorem 3] and Proposition 1, $S-$module $\mathrm{Hom}_R(S,M)$ is semisimple if and only if $\mathrm{Hom}_R(S,M)$ is semisimple as an $R-$module. On the other hand, we can see that $R-$module $\mathrm{Hom}_R(S,M)$ is semisimple if and only if $R-$module M is semisimple since $\mathrm{Hom}_R(S,M) \cong \oplus_{i=1}^n M^{\sigma_i^{-1}}$ as $R-$modules. The conclusion is now clear. □

Direct Summands of modules

Proposition 3 *If S is a free normalizing extension or R and N_R is a submodule of the $R-$module M_R, then*

(1) $N\otimes_R S|M\otimes_R S$ as $S-$module if and only if $N_R|M_R$.

(2) $Hom_R(S,N)|Hom_R(S,M)$ as $S-$modules if and only if $N_R|M_R$.

Proof If $N_R|M_R$, then there is an $R-$submodule T_R of M_R such that $N_R\oplus T_R = M_R$. Thus $(N\otimes_R S)\oplus(T\otimes_R S) = M\otimes_R S$ and $\mathrm{Hom}_R(S,N)\oplus\mathrm{Hom}_R(S,T) = \mathrm{Hom}_R(S,M)$. So $N\otimes_R S|M\otimes_R S$ and $\mathrm{Hom}_R(S,N)|\mathrm{Hom}_R(S,M)$.

Conversely, suppose that $\mathrm{Hom}_R(S,N)|\mathrm{Hom}_R(S,M)$.Then there is an $S-$submodule T of $\mathrm{Hom}_R(S,M)$, such that $\mathrm{Hom}_R(S,N)\oplus T = \mathrm{Hom}_R(S,M)$. Let $V = \{m \in M$: there exist $m_2,\cdots,m_n\} \in N$ and an $f\in T$ such that $f(1) = m$ and $f(a_i) = m_i$ for $i = 2,\cdots,n\}$. Obviously, V is an $R-$submodule of M and $V\cap N = 0$. For each $m\in M$, define $f\in \mathrm{Hom}_R(S,M)$ by $f(1) = m, f(a_i) = 0$, for $i = 2,\cdots,n$. Thus $f = f_1 + f_2$, where $f_1 \in \mathrm{Hom}_R(S,N), f_2\in T$. So $f_2(a_i) = -f_1(a_i) \in N$, for $i = 2,\cdots,n$. Hence $m = f_1(1)+f_2(1)$, where $f_1(1)\in N$ and $f_2(1)\in V$. Then $M = N\oplus V$ and $N_R|M_R$.

A similar argument can prove that $N\otimes_R S|M\otimes_R S$ implies $N_R|M_R$. □

Essential Submodules

In the following, the notation $N_R \xrightarrow{*} M_R$ means that the $R-$module N_R is an essential submodule of the $R-$module M_R. In [5] it has been proved that if S is a normalizing extension of R, then $N_R \xrightarrow{*} M_R$ implies $\mathrm{Hom}_R(S,N) \xrightarrow{*} \mathrm{Hom}_R(S,M)$ as $R-$modules and as $S-$modules.

Next we have the corresponding result for tensor products.

Proposition 4 *Let S be a normalizing extension of R, N_R be a submodule of M_R and the inclusion map $i : N \otimes_R S \to M \otimes_R S$ be a monomorphism, then $N_R \xrightarrow{*} M_R$ implies $N \otimes_R S \xrightarrow{*} M \otimes_R S$ as $R-$modules and consequently as $S-$modules.*

Proof Suppose that $0 \neq T$ is an $R-$submodule of $M \otimes_R S$, we show that $(N \otimes_R S) \cap T \neq 0$.

Let $V_i = \{m \in M :$ there exist $m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_n$ such that $m \otimes a_i + \sum_{j \neq i} m_j \otimes a_j \in T\}$, for $i = 1, 2, \cdots, n$. Obviously, each $V_i, i = 1, 2, \cdots, n$ is an $R-$submodule of M. Since $T \neq 0, V_i, i = 1, 2, \cdots, n$ are not all zero. Choose $V_{i_1} \neq 0$ with the least i_1, then $V_{i_1} \cap N \neq 0$. Take $T_1 = \{\sum_{i=1}^n m_i \otimes a_i \in T, m_{i_1} \in V_{i_1} \cap N\}$. Thus $0 \neq T_1 \subseteq T$ and it is an $R-$submodule of $M \otimes_R S$. Replace T by T_1, and consider $V_j^! = \{m \in M :$ there exist $m_{i_1}, \cdots, m_{j-1}, m_{j+1}, \cdots, m_n$ such that $m \otimes a_j + \sum_{k \neq j} m_k \otimes a_k \in T_1\}$ for each $j \geq i_1 + 1$. If $V_j^! = 0$, for all $j \geq i_1 + 1$, then $T_1 = (V_{i_1} \cap N) \otimes a_i \subseteq N \otimes_R S$ and $(N \otimes_R S) \cap T \neq 0$. Otherwise, choose $V_{i_2}^! \neq 0$ with the least i_2. Then $V_{i_2}^! \cap N \neq 0$ and we take $T_2 = \{\sum_{i=1}^n m_i \otimes a_i \in T_i, m_{i_2} \in V_{i_2}^! \cap N\}$. Thus $0 \neq T_2 \subseteq T_1$ and it is an $R-$submodule of $M \otimes_R S$.

Going on in this way, we can get $0 \neq T_l \subseteq T$ and $T_l \subseteq N \otimes_R S$, so that $(N \otimes_R S) \cap T \neq 0$. Therefore, $N \otimes_R S \xrightarrow{*} M \otimes_R S$ as $R-$modules and as $S-$modules.

Proposition 5 *Let S be a free normalizing extension of R and let N_R be a submodule of M_R. Then $N_R \xrightarrow{*} M_R$ if $N \otimes_R S \xrightarrow{*} M \otimes_R S$ or if $Hom_R(S, N) \xrightarrow{*} Hom_R(S, M)$ as $S-$modules.*

Proof If $N \otimes_R S \xrightarrow{*} M \otimes_R S$ as $S-$modules, then $N \otimes_R S \xrightarrow{*} M \otimes_R S$ as $R-$modules by [4, Proposition 1.1]. But $N \otimes_R S \cong \oplus_{i=1}^n N^{\sigma_i}$ and $M \otimes_R S \cong M^{\sigma_i}$ as $R-$modules. Thus we have $N_R \xrightarrow{*} M_R$. A similar argument can be used in the case of $\mathrm{Hom}_R(S, N) \xrightarrow{*} \mathrm{Hom}_R(S, M)$.

Injective modules

For injective modules it is known that if S is a normalizing extension of R and M is an $R-$module, then $\mathrm{Hom}_R(S, M)$ is $S-$injective if and only if M is $R-$injective [5, Corollary 2]. Here we consider the other cases.

Proposition 6 *Let S be a free normalizing extension of R. Then (1) $S-$module M_S $S-$injective implies M_R $R-$injective, (2) if M is an $R-$module, $M \otimes_R S$ $S-$injective implies M_R $R-$injective.*

Proof (1) If M_S is $S-$injective, then M_S is isomorphic to a direct summand of an $S-$module of the form $\mathrm{Hom}_z(S, D)$ with D a divisible Abelian group by [3, Corollary 5.5.4]. But $\mathrm{Hom}_z(S, D) \cong \oplus_{i=1}^n \mathrm{Hom}_z(Ra_i, D)$ as $R-$modules. Because each $\mathrm{Hom}_z(Ra_i, D) \cong \mathrm{Hom}_z(R, D)$ is Rinjective, it follows that $\mathrm{Hom}_z(S, D)$ is $R-$injective. Hence M_R, a direct summand of the $R-$module $\mathrm{Hom}_z(s, D)$, is $R-$injective.

(2) For every $R-$monomorphism $f : M_R \to B_R$, the map $f \otimes 1 : M \otimes_R S \to B \otimes_R S$ is an $S-$monomorphism since S is a free $R-$module. If $M \otimes_R S$ is $S-$injective, then $\mathrm{Im}(f \otimes 1) | B \otimes_R S$, that is, $\mathrm{Im} f \otimes S | B \otimes_R S$. Thus $\mathrm{Imf} | B_R$ by Proposition 3. This proves that M_R is $R-$injective. □

Propposition 7 *Let S be an excellent extension of R. (1) If M is an $S-$module, then M_R $R-$injective implies M_S $S-$injective, (2) $R-$module M is $R-$injective implies $S-$module $M \otimes_R S$ be $S-$injective.*

Proof (1) In order to prove that M_S is $S-$injective, we only need show that for every $S-$monomorphism $f : M_S \to B_S$, $(\mathrm{Imf})_S | B_S$. In fact, f is also an $R-$monomorphism, so $(\mathrm{Imf})_R | B_R$ since M_R is $R-$injective. Hence $(\mathrm{Imf})_S | B_S$ because S is an excellent extension of R.

(2) $M \otimes_R S \cong \oplus_{i=1}^n M^{\sigma_i}$ as $R-$modules. Since M_R is $R-$injective, each M^{σ_i} is $R-$injecitve. Thus $M \otimes_R S$ is $R-$injective and $S-$injective by (1). □

Projective modules

Proposition 8 *Let S be a free mormalizing extension of R and M be and $S-$module, then M is $S-$projective implies M be $R-$projective. If S is an excellent extension of R, then $S-$module M $R-$projective implies M $S-$projective.*

Proof Suppose that M_S is $S-$projective. Then M_S is a direct summand of a free $S-$module. Since S is a free $R-$module, M_R is a direct summand of a free $R-$module and so M_R is $R-$projective.

Suppose that $S-$module M is $R-$projective. Let $f : A_S \to M_S$ be an $S-$epimorphism. Then f is also an $R-$epimorphism. Thus $(\mathrm{Ker} f)_R | A_R$, and therefore, $(\mathrm{Ker} f)_S | A_S$ since S is an excellent extension of R. This proves that M_S is $S-$projective. □

As is well known, if R is subring of S and $R-$module M is $R-$projective, then $M \otimes_R S$ is $S-$projective.

Using Proposition 8, the following proposition can be obtained immediately.

Proposition 9 *If S is an excellent extension of R and M is a projective $R-$module, then $Hom_R(S, P)$ is $S-$projective.*

Proposition 10 *Let S be a free normalizing extension of R and let M be an $R-$module.*

(1) $M \otimes_R S$ $S-$projective implies M $R-$projective.

(2) $Hom_R(S, M)$ $S-$projective implies M $R-$projective.

Notice that in these cases, $M \otimes_R S \cong \oplus_{i=1}^n M^{\sigma_i}$ and $\mathrm{Hom}_R(S, M) \cong \oplus_{i=1}^n M^{\sigma_i}$.

Flat modules

It is easy to see that when R is a subring of S and $R-$module M is an $R-$flat module, then $S-$module $M \otimes_R S$ is an $S-$flat module. On the other hand, we have the following.

Proposition 11 *If S is a free normalizing extension of R and M is an $R-$module, then $M \otimes_R S$ $S-$flat implies M $R-$flat.*

Proof Consider a left $R-$monomorphism $f :_R A \to_R B$. Since S is a free $R-$module, $1 \otimes f : S \otimes_R A \to S \otimes_R B$ is a left $S-$monomorphism. So, $1 \otimes (1 \otimes f) : (M \otimes_R S) \otimes_S (S \otimes_R A) \to (M \otimes_R S) \otimes_S (S \otimes_R B)$ is a monomorphism. But $(M \otimes_R S) \otimes_S (S \otimes_R A) \cong (M \otimes_R S) \otimes_R A$, and $(M \otimes_R S) \otimes_S (S \otimes_R B) \cong (M \otimes_R S) \otimes_R B$. So we have $1 \otimes f : (M \otimes_R S) \otimes_R A \to (M \otimes_R S) \otimes_R B$

is a monomorphism. Since $M_R \cong (M \otimes_R 1)_R \subseteq M \otimes_R S$, $1 \otimes f : M \otimes_R A \to M \otimes_R B$ is a monomorphism. Hence M is an $R-$flat module. □

Proposition 12 *Let S be an excellent extension of R.*

(1) $S-$module M is $S-$flat if and only if M is $R-$flat.

(2) $R-$module M is $R-$flat if and only if $Hom_R(S, M)$ is $S-$flat.

Proof (1) If M is $S-$flat for every left $R-$monomorphsim $f : {}_R A \to {}_R B$, $1 \otimes f : S \otimes_R A \to S \otimes_R B$ is a left $S-$monomorphism. Thus, $1 \otimes (1 \otimes f) : M \otimes_S (S \otimes_R A) \to M \otimes_S (S \otimes_R B)$ is monomorphism. So, $1 \otimes f : M \otimes_R A \to M \otimes_R B$ is a monomorphism. Therefore, M is $R-$flat. (Remark: Here is only needed that S is a free normalizing extension of R).

Conversely, if M is $R-$flat, then $M \otimes_R S$ is $S-$flat. According to [2, Lemma 1], $M_S | (M \otimes_R S)_S$, therefore, M_S is $S-$flat.

(2) Since $\mathrm{Hom}_R(S, M) \cong \oplus_{i=1}^n M^{\sigma_i^{-1}}$ as $R-$modules. $\mathrm{Hom}_R(S, M)$ is $R-$flat if and only if M is $R-$flat. But $\mathrm{Hom}_R(S, M)$ is $S-$flat if and only if $\mathrm{Hom}_R(S, M)$ is $R-$flat. The conclusion is clear. □

Projective covers

If M is an $R-$module, an $R-$epimorphism $\xi : P_R \to M_R$ is called a projective cover of M if P_R is projective and $\mathrm{Ker}\xi$ is small in M_R, the latter is denoted by $\mathrm{Ker}\xi \xrightarrow{0} M_R$.

Proposition 13 *Let S be a free normalizing extension of R and M be an $R-$module. If M_R has a projective cover $\xi : P_R \to M_R$ then $(M \otimes_R S)_S$ has a projective cover $\xi \otimes 1 : P \otimes_R S \to M \otimes_R S$.*

Proof Obviously, $P \otimes_R S$ is $S-$projective and $\xi \otimes 1$ is an $S-$epimorphism. It remains to show that $\mathrm{Ker}(\xi \otimes 1) \xrightarrow{\circ} P \otimes_R S$. Since $P \otimes_R S = \oplus_{i=1}^n (P \otimes a_i)$ and $M \otimes_R S = \oplus_{i=1}^n (M \otimes a_i)$, it is easy to see that $\mathrm{Ker}(\xi \otimes 1) = \oplus_{i=1}^n (\mathrm{Ker}\xi \otimes a_i)$.

Suppose that $\mathrm{Ker}(\xi \otimes 1) + T = P \otimes_R S$, where T is an $S-$submodule of $P \otimes_R S$. Then $(\mathrm{Ker}\xi \otimes a_i) + (\oplus_{i=1}^n (\mathrm{Ker}\xi \otimes a_i) + T) = \oplus_{i=1}^n (P \otimes a_i)$. We have that $\oplus_{i=1}^n (\mathrm{Ker}\xi \otimes a_i) + T \supseteq P \otimes a_i$, since $\mathrm{Ker}\xi \xrightarrow{\circ} P_R$. Furthermore, $\oplus_{i=2}^n (\mathrm{Ker}\xi \otimes a_i) + T = \oplus_{i=2}^n (\mathrm{Ker}\xi \otimes a_i)$.

Going on in this way, finally, we have $T = \oplus_{i=1}^n (P \otimes a_i) = P \otimes_R S$. This shows that $\mathrm{Ker}(\xi \otimes 1) \xrightarrow{\circ} P \otimes_R S$ and completes the proof. □

Proposition 14 *Let S be an excellent extension of R and let M be an $R-$module. If $R-$module M has a projective cover $\xi : P_R \to M_R$, then $S-$module $Hom_R(S, M)$ has a projective cover $Hom_R(1, \xi) : Hom_R(S, P) \to Hom_R(S, M)$.*

Proof Since S is an excellent extension of R, P $R-$projective implies $\mathrm{Hom}_R(S, P)$ $S-$projective by Proposition 9. For every $h \in \mathrm{Hom}_R(S, M)$. Let $h(a_i) = m_i \in M, i = 1, 2, \cdots, n$. Take $p_i \in P, i = 1, 2, \cdots, n$ such that $\xi(p_i) = m_i, i = 1, 2, \cdots, n$ and construct $g \in \mathrm{Hom}_R(S, P)$ by $g(a_i) = p_i, i = 1, 2, \cdots, n$. Then $\mathrm{Hom}(1, \xi)(g) = \xi g = h$. Therefore, $\mathrm{Hom}(1, \xi)$ is an epimorphism. It remains to show that $\mathrm{Ker}(\mathrm{Hom}(1, \xi)) \xrightarrow{\circ} \mathrm{Hom}_R(S, P)$

In fact, $\mathrm{Ker}(\mathrm{Hom}(1, \xi)) = \{g \in \mathrm{Hom}_R(S, P) : g(a_i) \subseteq \mathrm{Ker}\xi, i = 1, 2, \cdots, n\}$. Let $\mathrm{Ker}_i(\mathrm{Hom}(1, \xi) = \{g \in \mathrm{Hom}_R(S, P) : g(a_i) \subseteq \mathrm{Ker}\xi, g(a_i) = 0$ for all $j \neq i\}$. Obviously,

$\mathrm{Ker}(\mathrm{Hom}(1,\xi)) = \sum_{i=1}^{n} \mathrm{Ker}_i(\mathrm{Hom}(1,\xi))$ and each $\mathrm{Ker}_i(\mathrm{Hom}(1,\xi))$ is an R-module. Now, we prove that each $\mathrm{Ker}_i(\mathrm{Hom}(1,\xi)) \xrightarrow{0} \mathrm{Hom}_R(S,P)$ as R-modules.

Suppose that $\mathrm{Ker}_i(\mathrm{Hom}(1,\xi)) + T = \mathrm{Hom}_R(S,P)$, where T is an R-module of $\mathrm{Hom}_R(S,P)$. We have to show that $T = \mathrm{Hom}_R(S,P)$. Let $\mathrm{Hom}_R(S,P)_i = \{g \in \mathrm{Hom}_R(S,P) : g(a_j) = 0, j \neq i\}$, and $T_i = \{g \in T : g(a_j) = 0, j \neq i\}$. Then $\mathrm{Ker}_i(\mathrm{Hom}(1,\xi)) + T_i = \mathrm{Hom}_R(S,P)_i$, and hence $\mathrm{Ker}\xi + T_i(a_i) = P$, where $T_i(a_i) = \{g(a_i) : g \in T_i\}$. Obviously, $T_i(a_i)$ is an R-submodule of P_R and so $T_i(a_i) = P$ since $\mathrm{Ker}\xi \xrightarrow{\circ} P$. Therefore $T_i = \mathrm{Hom}_R(S,P)_i$ and $T = \mathrm{Hom}_R(S,P)$.

Because each $\mathrm{Ker}_i(\mathrm{Hom}(1,\xi)) \xrightarrow{\circ} \mathrm{Hom}_R(S,P)$, it follows that $\mathrm{Ker}(\mathrm{Hom}(1,\xi)) \xrightarrow{\circ} \mathrm{Hom}_R(S,P)$ as R-modules and as S-modules. This proves that $\mathrm{Hom}(1,\xi) : \mathrm{Hom}_R(S,P) \to \mathrm{Hom}_R(S,M)$ is a projective cover of $\mathrm{Hom}_R(S,M)$. □

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摘 要

设环 S 是环 R 的正规化扩张.本文讨论了 R 一摸与 S 一摸两者间的若干相关性质.