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Certain Class of Prestarlike Functions[*1]

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Abstract We investigate a new class of functions for which we obtain, representation theorem, coefficient inequalities, distortion theorem, closure theorem, extreme points and radius of convexity theorem.

1. Introduction

Let S_p denote the class of fucntions $f(z) = z^p \sum_{m=2}^{\infty} a_{p+m-1} z^{p+m-1}$ analytic and $p-$valent in the unit disc $E = \{z \in \mathbf{C} : |z| < 1\}$.

A function f will be called $p-$valent or multivalent in E if no value of the function is taken for more than p values of z in E and if at least one value of the function is taken on exactly p times. When p is one, the function will be called univalent or simple.

A function f in S_p is said to be subordinate to a function F, denoted by $f \prec F$, if there exists an analytic function $\Phi(z)$ with $|\Phi(z)| < |z|, z \in E$ such that $f = F \circ \Phi$.

For $0 \le \alpha < 1, p \ge 1$ we say that $f \in S_p^*(\alpha)$ if and only if

$$\frac{zf'(z)}{f(z)} \prec p \cdot \frac{1+(2\alpha-1)z}{1+z}, z \in E,$$

or equivalently $f \in S_p^*(\alpha)$ if and only if

$$\left| \frac{\frac{zf'(z)}{f(z)} - p}{\frac{zf'(z)}{f(z)} - (2\alpha-1)p} \right| < 1 \quad \text{for } z \in E.$$

Consider the function $s_\alpha^p(z) = z^p/(1-z)^{2p(1-\alpha)}$ which is equivalent to

$$z^p + \sum_{m=2}^{\infty} C(\alpha, m) z^{p+m-1}, \tag{1}$$

where

$$C(\alpha, m) = \frac{\prod_{k=2}^{m}(2p(1-\alpha)+k-2)}{(m-1)!}. \tag{2}$$

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Clearly, $s_\alpha^p(z) \in S_p^*(\alpha)$ and it is a decreasing function of α.

The convolution or Hadamard product of two power series

$$f(z) = \sum_{m=1}^{\infty} a_{p+m-1} z^{p+m-1},$$

and $g(z) = \sum_{m=1}^{\infty} b_{p+m-1} z^{p+m-1}$ is defined as the power series

$$(f * g)(z) = \sum_{m=1}^{\infty} a_{p+m-1} b_{p+m-1} z^{p+m-1}.$$

An analytic and $p-$valent function of the form $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$ is said to be in the class $P_\alpha^p, p \geq 1, 0 \leq \alpha < 1$, if $f * S_\alpha^p \in s_p^*(\alpha)$, where $s_\alpha^p(z)$ is defined as in (1).

The class $R_\alpha^1 \equiv R_\alpha$, class of prestarlike functions, was introduced by Ruscheweyh St. [2] in 1977 and further studied by Sheil-Small, Silverman and Silvia [3].

A function f in R_α^p is said to be in $R_p[\alpha]$ if it can be expressed in the form

$$f(z) = z^p - \sum_{m=2}^{\infty} |a_{p+m-1}| z^{p+m-1}.$$

2. Representation theorem

Theorem 1 *A necessary and sufficient condition for f to be in the class $R^p[\alpha]$ is that the functional*

$$G(\alpha, p, z) = \frac{f(z) * \{s_\alpha^p(z)/(1-z)\}}{f(z) * s_\alpha^p(z)}$$

satisfy $Re\{G(\alpha, p, z)\} > \frac{1}{2p(1-\alpha)}[p(1-2\alpha)+\alpha]$ for $z \in E$.

Proof Suppose $f \in R^p[\alpha]$, then $g = s_\alpha^p * f \in S_p^*(\alpha)$. From the identity

$$\frac{s_\alpha^p(z)}{1-z} = s_\alpha(z)^p * \frac{1}{2p(1-\alpha)}[p(1-2\alpha)h(z) + zh'(z)]$$

where $h(z) = z^p/(1-z)$, it follows that

$$\begin{aligned} f(z) * \frac{s_\alpha^p(z)}{1-z} &= f(z) * s_\alpha^p(z) * \frac{1}{2p(1-\alpha)}[p(1-2\alpha)h(z) + zh'(z)] \\ &= g(z) * \frac{1}{2p(1-\alpha)}[[p(1-2\alpha)h(z) + zh'(z)] \\ &= \frac{1}{2p(1-\alpha)}[p(1-2\alpha)g(z) + zg'(z)]. \end{aligned}$$

It follows that

$$\frac{f(z) * \frac{s_\alpha^p(z)}{1-z}}{f(z) * s_\alpha^p(z)} = \frac{1}{2p(1-\alpha)}[p(1-2\alpha) + \frac{zg'(z)}{g(z)}]. \tag{3}$$

This proves the necessary part of the theorem. Conversely, (3) implies that $g \in S_p^*(\alpha)$ and hence the theorem.

The following result which is due to R.M. Goel and N.S. Sohi [1] is used for the particular values of $A = 1$ and $B = 2\alpha - 1$ in the next section.

Theorem 1[1] *A function $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n}$ belongs to $T_p^*(A, B)$ if and only if*

$$\sum_{n=1}^{\infty} [(1 + A)n + (B - A)p] |a_{p+n}| \le (B - A)p.$$

3. The result is sharp.

Coefficient inequalities

Some necessary and sufficient conditions for a function to be in $R^p[\alpha]$ are given by the following

Theorem 2 *A function $f(z) = z^p - \sum_{m=2}^{\infty} |a_{p+m-1}| z^{p+m-1}$ is in $R^p[\alpha]$, $0 \le \alpha < 1$ if and only if*

$$\sum_{m=2}^{\infty} [(m - 1) + (1 - \alpha)p] C(\alpha, m) |a_{p+m-1}| \le (1 - \alpha)p.$$

The result is sharp for the function

$$f(z) = z^p - \frac{(1 - \alpha)p}{[m - 1 + (1 - \alpha)p] C(\alpha, m)} z^{p+m-1} \text{ for } m \ge 2.$$

Proof We know from [1] that $\sum_{m=2}^{\infty} [m - 1 + (1 - \alpha)p] |a_{p+m-1}| \le (1 - \alpha)p$ if and only if $f(z) = z^p - \sum_{m=2}^{\infty} |a_{p+m-1}| z^{p+m-1}$ belongs to $s_p^*(\alpha)$. Since $f * s_\alpha^p(z) = z^p - \sum_{m=2}^{\infty} |a_{p+m-1}| C(\alpha, m) z^{p+m-1}$, the result follows.

Corollary *If $f(z) = z^p - \sum_{m=2}^{\infty} |a_{p+m-1}| C(\alpha, m) z^{p+m-1} \in R^p[\alpha], 0 \le \alpha < 1, p \ge 1$, then $|a_{p+m-1}| \le \frac{(1-\alpha)p}{[m-1+(1-\alpha)p] C(\alpha,m)}$, with equality only for function of the form*

$$f(z) = z^p - \frac{(1 - \alpha)p}{[m - 1 + (1 - \alpha)p] C(\alpha, m)} z^{p+m-1}, m \ge 2.$$

4. Extreme points

Theorem 3 *Set $f_p(z) = z^p$ and $f_{p+m-1}(z) = z^p - \frac{(1-\alpha)p}{[m-1+(1-\alpha)p] C(\alpha,m)} z^{p+m-1}$. Then $f \in R^p[\alpha]$, $0 \le \alpha < 1, p \ge 1$ if and only if it can be expressed in the form $\sum_{m=1}^{\infty} \lambda_{p+m-1} f_{p+m-1}(z)$ where $\lambda_{p+m-1} \ge 0$ and $\sum_{m=1}^{\infty} \lambda_{p+m-1} = 1$.*

Proof Suppose $f(z) = \sum_{m=1}^{\infty} \lambda_{p+m-1} f_{p+m-1}(z)$. Then

$$\sum_{m=2}^{\infty} \frac{(1 - \alpha)p \lambda_{p+m-1}}{[m - 1 + (1 - \alpha)p] C(\alpha, m)} \cdot \frac{[m - 1 + (1 - \alpha)p] C(\alpha, m)}{(1 - \alpha)p} = \sum_{m=2}^{\infty} \lambda_{p+m-1} = 1 - \lambda_p \le 1.$$

Therefore, $f \in R^p[\alpha]$ by Theorem 2. Conversely, suppose

$$f(z) = z^p - \sum_{m=2}^{\infty} |a_{p+m-1}| z^{p+m-1} \in R^p[\alpha], 0 \le \alpha < 1, p \ge 1.$$

Then

$$|a_{p+m-1}| \le \frac{(1-\alpha)p}{[m-1+(1-\alpha)p]C(\alpha,m)}, m \ge 2.$$

Set

$$\lambda_{p+m-1} = \frac{[m-1+(1-\alpha)p]C(\alpha,m)a_{p+m-1}}{(1-\alpha)p} \text{ and } \lambda_p = 1 - \sum_{m=2}^{\infty} \lambda_{p+m-1}.$$

From Theorem 2, it follows that $\lambda_p \ge 0$. Since $f(z) = \sum_{m=1}^{\infty} \lambda_{p+m-1} f_{p+m-1}(z)$, the proof is complete.

5. Distortion theorems

Theorem 4 *If* $f(z) = z^p - \sum_{m=2}^{\infty} |a_{p+m-1}| z^{p+m-1} \in R^p[\alpha], 0 \le \alpha < 1, p \ge 1$. *Then*

(i)
$$r^p - \frac{r^{p+1}}{2(1+(1-\alpha)p)} \le |f(r)| \le r^p + \frac{r^{p+1}}{2(1+(1-\alpha)p)}, |z| = r,$$

(ii)
$$pr^{p-1} - \frac{(p+1)r^p}{2(1+(1-\alpha)p)} \le |f'(r)| \le pr^{p-1} + \frac{(p+1)r^p}{2(1+(1-\alpha)p)}, |z| = r.$$

The result is sharp.

Proof From Theorem 3, we have that

$$r^p - \max_m \frac{(1-\alpha)p}{[m-1(1-\alpha)p]C(\alpha,m)} \le |f(r)| \le r^p + \max_m \frac{(1-\alpha)p}{[m-1(1-\alpha)p]C(\alpha,m)} r^{p+m-1},$$

where $m \ge 2, |z| = r$. It suffices to show that $A(\alpha,m,p) = \frac{(1-\alpha)p}{[m-1+(1-\alpha)p]C(\alpha,m)}$ is a decreasing function of m. From (2) we have that $C(\alpha,m) = [2p(1-\alpha)+m-1]/mC(\alpha,m)$. Therefore, we have $A(\alpha,m,p) \ge A(\alpha,m+1,p), m \ge 2$ whenever

$$\frac{[m+(1-\alpha)p][2p(1-\alpha)+m-1]}{m} \ge [m-1+(1-\alpha)p].$$

This is equivalent to $pm(1-\alpha) + 2p^2(1-\alpha)^2 + p(m-1)(1-\alpha) \ge 0$, which is true and hence proves the first part of the theorem.

Similarly, one can get the inequality (ii).

Corollary *Let* $f \in R^p[\alpha]$. *Then the disc* $|z| < 1$ *is mapped onto a domain that contains the disc* $|w| < \frac{1+2(1-\alpha)p}{2(1+(1-\alpha)p)}$.

The result follows upon letting $r \to 1$ in the equality (i) on the left side of Theorem 4.

6. Clourse Theorem

We prove that $R^p[\alpha]$ is closed under convex linear combination.

Theorem 5 *If* $f(z) = z^p - \sum_{m=2}^{\infty} |a_{p+m-1}| C(\alpha,m) z^{p+m-1}$ *and* $g(z) = z^p - \sum_{m=2}^{\infty} |b_{p+m-1}| C(\alpha,m) z^{p+m-1}$ *are in* $R^p[\alpha]$, *then* $h(z) = z^p - \sum_{m=2}^{\infty} \frac{1}{2}\{|a_{p+m-1}| + |b_{p+m-1}|\} z^{p+m-1}$ *is also in* $R^p[\alpha]$.

Proof Since f and g are in $R^p[\alpha]$ we have by Theorem 2

$$\sum_{m=2}^{\infty} [m-1+(1-\alpha)p] C(\alpha,m) |a_{p+m-1}| \leq (1-\alpha)p,$$

and

$$\sum_{m=2}^{\infty} [m-1+(1-\alpha)p] C(\alpha,m) |b_{p+m-1}| \leq (1-\alpha)p,$$

Adding these two equalitites we get

$$\sum_{m=2}^{\infty} [m-1] + (1-\alpha)p] C(\alpha,m) \frac{1}{2}\{|a_{p+m-1}| + |b_{p+m-1}|\} \leq (1-\alpha)p,$$

which implies that $h(z)$ belongs to $R^p[\alpha]$.

7. Radius of convexity theorem

Theorem 6 *If* $f(z) \in R^p[\alpha]$ *then* $f(z)$ *is* $p-$*valent convex in the disc*

$$|z| < K_p = \inf_m \left[\{\frac{m-1+(1-\alpha)p}{(1-\alpha)p} [\frac{p}{p+m-1}\}^{2\frac{1}{m-1}} \right]$$

The result is sharp.

Proof It is sufficient to prove that $|1 + \frac{zf''(z)}{f'(z)} - p| \leq p$ for $|z| < K_p$. We have

$$\begin{aligned} |1 + \frac{zf''(z)}{f'(z)} - p| &= |\frac{\sum_{m=2}^{\infty}(p+m-1)(m-1)|a_{p+m-1}|z^{m-1}}{p - \sum_{m=2}^{\infty}(p+m-1)|a_{p+m-1}|z^{m-1}}| \leq p \\ &\leq \sum_{m=2}^{\infty} (p+m-1)^2 |a_{p+m-1}||z|^{m-1} \leq p^2. \end{aligned} \tag{4}$$

But by Theorem 2, $\sum_{m=2}^{\infty} \frac{m-1+(1-\alpha)p}{(1-\alpha)p} C(\alpha,m)|a_{p+m-1}| \leq 1$. Hence (4) will be satisfied if

$$[\frac{p+m-1}{p}]^2 |z|^{m-1} \leq \frac{m-1+(1-\alpha)p}{(1-\alpha)p} C(\alpha,m),$$

or if

$$|z| \leq \{\frac{m-1+(1-\alpha)p}{(1-\alpha)p} [\frac{p}{p+m-1}]^2\}^{\frac{1}{m-1}}. \tag{5}$$

The theorem follows easily from (5).

References

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[3] T. Sheil Small, H. Silverman and E.M. Silvia,*Convolution multipliers and starlike functions*, J. Anal. Math.,**41**(1982), 181–192.