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The Stability of a Class of Difference Equations*

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Abstract. In this paper, we study the stability of a class of difference equations, obtain the sufficient conditions of asymptotic stability and uniformly asymptotic stability by applying Liapunov functions. These results are a extension of Wang Lian et al's results.

1. Introduction

Consider the difference equation

$$\begin{cases} x(t+1) = f(t,x) + g(t,y), \\ y(t+1) = h(t,x,y), \end{cases} \tag{1}$$

where $(t,x,y) \in I \times \mathbf{R}^n \times \mathbf{R}^m$, $I = \{t_0 + s : s = 0,1,2,\cdots\}$, we suppose that $f(t,x)+g(t,y) = x$ and $h(t,x,y) = y$ if and only if $x = 0$ and $y = 0$. Assume that functions f, g and h are denoted on $I \times R^n \times R^m$ such that solution $(x(t), y(t))$ of (1) which passes through point $(t_0, x_0, y_0) \in I \times R^n \times R^m$ has definition on I.

Definition 1. If a continuous funxtion $a : [0, r_1] \to R^+ = [0, \infty)$ (or $a : R^+ \to R^+$) satisfies $a(0) = 0$ and increasing on $[0, r_1]$(or on R^+), then call $a \in K$. Moreover, if $a : R^+ \to R^+, a \in K$ and $\lim_{r\to\infty} a(r) = \infty$, then call $a \in KR$.

Definition 2 If function $f(t,x) : I \times R^n \to R$ is continuous in $x \in R^n_H = \{x \in R^n, |x| \leq H\}$ and $\sum_{S=t_0}^{\infty} \max_{|x|\leq H} f(s,x) < \infty$,then call $f \in r^*$.

Definition 3. Function $V(t,x,y) : I \times R^n \times R^m \to R$ is a Liapunov function of (1) if V is continuous in (x,y) and $\Delta V(t,x,y) = V(t+1, x(t+1), y(t+1)) - V(t,x,y) \leq 0$.

Definition 4. If function $p(t) : I \to R^+$ satisfies for any a infinite sequence $I_1 \subset I, \sum_{s\in I_1} p(s) = \infty$, then call $p \in IP$.

Lemma $p(t) \in IP$ *if and only if for arbitrary* $c > 0$ *there is* $K(c) \in N^+ = \{0,1,2,\ldots\}$ *for any finite sequence* $\{t_1, t_2, \ldots, t_{k(c)}\} \subset I$, $\sum_{i=1}^{k(c)} p(t_i) \geq c$.

Proof If there is $c_0 > 0$ such that for any $n \in N^+$ there is $\{t_1, t_2, \ldots, t_n\} \subset I$ and $\sum_{i=1}^{n} p(t_i) < c_0$. Then let $n \to \infty$we obtain $\sum_{i=1}^{\infty} p(t_i) \leq c_0$,this is a contradiction.

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Let $c = n$, then there is $K(n)$ such that $\sum_{i=1}^{n} p(t_i) \geq n$, let $n \to \infty$, we obtain $\sum_{i=1}^{\infty} p(t_i) = +\infty$, hence $p \in IP$.

2. Main theroems

We have the following theorems.

Theorem 1 *Suppose that (1) satisfies*

1). There is a liapunov function $V_1(t,x,y)$ *on* $I \times R_H^n \times R_H^m$ *such that* $V_1(t,0,0) = 0, V_1(t,x,y) \geq a_1(|\ x\ | + |\ y\ |)$ *and* $\Delta V_{1(1)}(t,x,y) \leq -p_1(t)c_1(|\ y\ |) + u_1(t,x,y) + h_1(t,x,y)V_1(t,x,y) \leq 0$.

2). For equation

$$x(t+1) = f(t,x), \tag{2}$$

there is a Liapunov function $V_2(t,x)$ *on* $I \times R_H^n$ *such that* $V_2(t,x) \geq a_2(|\ x\ |), V_2(t,0) = 0$ *and* $\Delta V_{2(2)}(t,x) \leq -p_2(t)c_2(V_2(t,x)) + u_2(t,x) + h_2(t,x)V_2(t,x) \leq 0$,

Where $a_1, a_2, c_1, c_2 \in K, p_1, p_2 \in IP, u_1, u_2, h_1, h_2 \in r^*$ *and* $h_1, h_2 \geq 0$.*Then solution* $x = 0, y = 0$ *of (1) is asymptotically stable.*

Proof From the condition 1), we obtain zero solution is stable, therefore, for arbitrary $t_0 \in I$ there is $\delta(t_0) > 0$ such that $|\ x(t)\ | + |\ y(t)\ | \leq H$ as $|\ x_0\ | + |\ y_0\ | \leq \delta(t_0)$, where $x(t) = x(t,x_0,y_0), y(t) = y(t,x_0,y_0)$. First of all we prove $y(t) \to 0$ as $t \to \infty$. If there is $\varepsilon_0 > 0$ and $T > t_0$ such that $|\ y(t)\ | \geq \varepsilon_0$ as $t \geq T$, then by condition 1), we obtain

$$\begin{aligned} V_1(t,x(t),y(t)) &\leq V_1(T,x(T),y(T)) - \sum_{s=T}^{t-1} p_1(s)c_1(|\ y(s)\ |) \\ &+ \sum_{s=T}^{t-1}(u_1(s,x(s),y(s)) + h_1(s,x(s),y(s))V_1(s,x(s),y(s))). \end{aligned} \tag{3}$$

Since $V(t,x(t),y(t)) \leq V(t_0,x_0,y_0)$ and $u_1, h_1 \in r^*$, we can obtain

$$\begin{aligned} &V(t,x(t),y(t)) \leq V_1(T,x(T),y(T)) - c_1(\varepsilon_0)\sum_{s=T}^{t-1} p_1(s) \\ &+ \sum_{s=T}^{t=1} \max_{|x|+|y|\leq H} u_1(s,x(s),y(s)) + V(t_0,x_0,y_0)\sum_{s=T}^{t-1} \max_{|x|+|y|\leq H} h_1(s,x(s),y(s)). \end{aligned} \tag{4}$$

Since $p_1 \in IP$, hence we have $V_1(t,x(t),y(t)) \to -\infty$ as $t \to \infty$.Therefore $\liminf_{t\to\infty} |\ y(t)\ | = 0$. If there is $\varepsilon_0 > 0$ and a sequence $\{t_i\}, \lim_{i\to\infty} t_i = \infty$ such that $|\ y(t_i)\ | \geq \varepsilon_0$, then similar to the above same method we obtain

$$\begin{aligned} &V_1(t,x(t),y(t)) \leq V_1(t_0,x_0,y_0) - \sum_{s=t_0}^{t-1} p_1(s)c_1(|\ y(s)\ |) \\ &+ \sum_{s=t_0}^{t-1} \max u_1(s,x(s),y(s)) + V_1(t_0,x_0,y_0)\sum_{s=t_0}^{t-1} \max h_1(s,x(s),y(s)). \end{aligned}$$

Since

$$\sum_{s=t_0}^{t-1} p_1(s)c_1(|\,y(s)\,|) \geq \sum_{t_i \leq t-1} p_1(t_i)c_1(|\,y(t_i)\,|) \geq c_1(\varepsilon_0) \sum_{t_i \leq t-1} p_1(t_i)$$

and $\sum_{i=1}^{\infty} p_1(t_i) = \infty$, so we can obtain $V_1(t, x(t), y(t)) \to -\infty$ as $t \to \infty$. Therefore $\limsup_{t\to\infty} y(t) = 0$.

Next we prove $x(t) \to 0$ as $t \to \infty$. Let $V_2(t) = V_2(t, x(t))$, we have

$$\begin{aligned}
\Delta V_2(t)\,|_{(1)} &= V_2(t+1, f(t,x)+g(t,y)) - V_2(t,x) \\
&= \Delta V_2(t)\,|_{(2)} + V_2(t+1, f(t,x)+g(t,y)) - V_2(t+1, f(t,x)) \\
&\leq -p_2(t)c_2(V_2(t)) + u_2(t,x) + h_2(t,x)V_2(t) + N(H)\,|\,g(t,y)\,|\,.
\end{aligned}$$

If $\liminf_{t\to\infty} V_2(t) \neq 0$, then there are $\varepsilon_0 > 0$ and $T > t_0$ such that $V_2(t) \geq \varepsilon_0$ as $t \geq T$. Since $p(|\,y(t)\,|) \to 0$ as $t \to \infty$, choose T such that $p(|\,y(t)\,|) \leq 1$ as $t \geq T$, from $|\,g(t,y)\,| \leq \varphi(t)p(|\,y\,|)$, we can obtain

$$\begin{aligned}
V_2(t) &\leq V_2(T)\sum_{s=T}^{t-1} p_2(s)c_2(V_2(s)) + \sum_{s=T}^{t-1}(u_2(s,x) + h_2(s,x)V_2(s)) + N(H)\sum_{s=T}^{t-1}\varphi(s) \\
&\leq V_2(T) - c_2(\varepsilon_0)\sum_{s=T}^{t-1} p_2(s) + \sum_{s=T}^{t-1}\max_{|x|\leq H} u_2(s,x) + \\
&+ V_2(T)\sum_{s=T}^{t-1}\max_{|x|\leq H} h_2(s,x) + N(H)\sum_{s=T}^{t-1}\varphi(s).
\end{aligned}$$

Therefore we can obtain $V_2(t) \to -\infty$ as $t \to \infty$. If $\limsup_{t\to\infty} V_2(t) \neq 0$, there are $\varepsilon_0 > 0$ and a sequence $\{t_i\}, \lim_{i\to\infty} t_i = \infty$ such that $V_2(t_i) \geq \varepsilon_0$. Choose $T > t_0$ such that $p(|\,y(t)\,|) \leq 1$ as $t \geq T$, we have

$$\begin{aligned}
V_2(t) &\leq V_2(T) - \sum_{s=T}^{t-1} p_2(s)c_2(V_2(s)) + \sum_{s=T}^{t-1}\max_{|x|\leq H} u_2(s,x) \\
&+ N(H)\sum_{s=T}^{t-1}\varphi(s) + V_2(T)\sum_{s=T}^{t-1}\max_{|x|\leq H} h_2(s,x).
\end{aligned}$$

Since $\sum_{s=T}^{t-1} p_2(s)c_2(V_2(s)) \geq \sum_{t_i \leq t-1} p_2(t_i)c_2(V_2(t_i)) \geq \sum_{t_i \leq t-1} c_2(\varepsilon_0)p_2(t_i)$ and $\sum_{i=1}^{\infty} p_2(t_i) = \infty$, so we can obtain $V_2(t) \to -\infty$ as $t \to \infty$, this is a contradiction. Therefore
$\limsup_{t\to\infty} V_2(t) = 0$. From $\lim_{t\to\infty} V_2(t) = 0$, we can obtain $\lim_{t\to\infty} x(t) = 0$, which completes the proof. □

From the proof of Theorem 1, we can estimate that domain D_{t_0}, $H = \{(x,y) \in R_H^n \times R_H^m : V_1(t_0, x, y) \leq a_1(H)\}$ be included in an asymptotical stable domain of(1).

Further we may obtain the sufficient condition of uniformly asymptotical stability of (1).

Theorem 2 *If equation (1) satisfies the conditions of Theorem 1, and*

3) $$u_1(t,x,y) \geq 0,\ u_2(t,x,y) \geq 0,$$

4) $V_1(t,x,y) \geq b_1(|x|+|y|), V_2(t,x) \geq b_2(|x|)$, *where* $b_1, b_2 \in K$,

then solution $x=0, y=0$ *of (1) is uniformly asymptotical stable.*

Proof In the first place, we easily obtain that solution $x=0, y=0$ is uniformly stable, therefore there is a $\delta_0 > 0$ such that $|x(t)|+|y(t)| \leq H$ as $|x_0|+|y_0| \leq \delta_0$.

For arbitrary $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $|x(t)|+|y(t)| < \varepsilon$ as $|x_0|+|y_0| < \delta$. Choose integer $K(c)$ such that p_1 and p_2 satisfy Lemma, and choose $\delta_1(\varepsilon) \in (0, \delta(\varepsilon)/2)$ such that $p(|y(t)|) \leq 1$ as $|y(t)| < \delta_1(\varepsilon)$. We can choose an even integer n such that

$$(1+N_1)b_1(H) - (n/2)c_1(\delta_1/2) + M_1 < 0,$$

$$(1+N_2)b_2(H) - nc_2(a_2(\delta/2)) + M_2 + N(H)K < 0,$$

where $M_1 = \sum_{s=t_0}^{\infty} \max_{|x|+|y|\leq H} u_1(s,x,y)$, $M_2 = \sum_{s=t_0}^{\infty} \max_{|x|+|y|\leq H} u_2(s,x,y)$, $K = \sum_{s=t_0}^{\infty} \varphi(s)$, $N_1 = \sum_{s=t_0}^{\infty} \max_{|x|+|y|\leq H} h_1(s,x,y)$, $N_2 = \sum_{s=t_0}^{\infty} \max_{|x|+|y|\leq H} h_2(s,x,y)$.

Let $E_i = \{t \in I : t_0 + 2(i-1)T_1 \leq t \leq t_0 + 2iT_1\}$ and $T_1 = nK(1)$, we will prove either there is $t_i' \in E_i$ such that $|x(t_i')|+|y(t_i')| < \delta(\varepsilon)$, or there is $t_i'' \in E_i$ such that $|y(t_i'')| \geq \delta_1(\varepsilon)$. Let $V_1(t) = V_1(t, x(t), y(t)), V_2(t) = V_2(t, x(t))$.

a) There is a $t_i^* \in [t_0 + 2(i-1)T_1, t_0 + 2(i-1)T_1 + T_1]$ such that $|y(t_i^*)| < \delta_1/2$. In fact, if $|y(t)| \geq \delta_1/2$ for $t \in [t_0+2(i-1)T_1, t_0+2(i-1)T_1+T_1]$, then we have

$$\begin{aligned}
& a_1(\delta_1/2) \leq V_1(t_0 + 2(i-1)T_1 + T_1) \\
\leq\ & V_2(t_0 + 2(i-1)T_1) - \sum_{s=t_0+2(i-1)T_1}^{t_0+2(i-1)T_1+T_1} p_1(s)c_1(|y(s)|) + M_1 + N_1V_1(t_0) \\
\leq\ & b_1(H) - c_1(\delta_1/2) \sum_{s=t_0+2(i-1)T_1}^{t_0+2(i-1)T_1+T_1} p_1(s) + M_1 + N_1b_1(H) \\
\leq\ & (1+N_1)b_1(H) - nc_1(\delta_1/2) + M_1 < 0,
\end{aligned}$$

this is a contradiction.

b) If for $t \in [t_i^*, t_0 + 2iT_1]$, we have $|y(t)| < \delta_1$, then there is $t_i' \in [t_i^*, t_0 + 2iT_1]$ such that $|x(t_i'| < \delta(\varepsilon)/2$, and $|x(t_i'|+|y(t_i')| < \delta(\varepsilon)$. In fact, if $|x(t)| \geq \delta(\varepsilon)/2$ for $t \in [t_i^*, t_0 + 2iT_1]$,then we have

$$\begin{aligned}
& a_2(\delta/2) \leq V_2(t_0 + 2iT_1) \\
\leq\ & V_2(t_0 + 2iT_1 - T_1) - \sum_{s=t_0+2iT_1-T_1}^{t_0+2iT_1} p_2(s)c_2(V_2(s)) + M_2 + N_2V_2(t_0) + N(H)K \\
\leq\ & (1+N_2)b_2(H) - nc_2(a_2(\delta/2)) + M_2 + N(H)K < 0,
\end{aligned}$$

this is a contradiction.

c) If the conclusion b) does not hold, then there is $t_i'' \in [t_i^*, t_0 + 2iT_1]$ such that $|y(t_i'')| \geq \delta_1(\varepsilon)$.

From the above discussion, we have either there is $i \leq T_1$ such that the conclusion b) holds, hence $|x(t)|+|y(t)| < \varepsilon$ as $t > t_0 + 2T_1^2$ or for every E_i there is $t_i'' \in E_i$

such that $| y(t_i'') | \geq \delta_1(\varepsilon)$. If the second result holds, then we can choose a finite sequence $\{t_j\} \subset \{t_i''\}$, $j = 1, 2, \cdots, n/2$, and $t_s \neq t_r$ for $s \neq r$, such that

$$\begin{aligned} V_1(t_0 + 2T_1^2) &\leq V_1(t_0) - \sum_{s=t_0}^{t_0+2T_1^2} p_1(s) c_1(| y(s) |) + M_1 + N_1 V_1(t_0) \\ &\leq (1 + N_1) b_1(H) - c_1(\delta_1/2) \sum_{j=1}^{n/2} p(t_j) + M_1 \\ &\leq (1 + N_1) b_1(H) - (n/2) c_1(\delta_1/2) + M_1 < 0, \end{aligned}$$

this is a contradiction. Therefore there must be $i \leq T_1$ and $t_i' \in E_i$ such that $| x(t_i') | + | y(t_i') | < \delta(\varepsilon)$. Thus we choose $T(\varepsilon) = 2T_1^2 = 2(nK(1))^2$ for $\varepsilon > 0$, then $| x(t) | + | y(t) | < \varepsilon$ as $t > t_0 + T(\varepsilon)$, so solution $x = 0, y = 0$ is uniformly asymptotical stable, theis completes the proof. □

In Theorems 1 and 2, if $x = 0$ and $p_1(t)$ =constant, then we can obtain the result in [1].

From Theorems 1 and 2, we can also obtain the following result.

Corollary *Suppose that there is Liapunov function $V(t, x)$ such that $V(t, 0) = 0, a(| x |) \leq V(t, x) \leq b(| x |)$ and $\Delta_{(2)} V(t, x) \leq -p(t) c(x)$, where $a, b, c \in K$ and $P \in IP$, then zerosolution $x = 0$ is uniformly asymptotical stable.*

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一类差分方程的渐近性

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摘　要

本文对高维非自治差分方程

$$\begin{cases} x(\tau+1) = f(\tau, x) + g(\tau, y) \\ y(\tau+1) = h(\tau, x, y) \end{cases} \tag{1}$$

得到了保证其平凡解渐近稳定，一致渐近稳定的判别准则，而对 V 函数的差分不要求负定性，推广了文[1]中相应的结果.