Journal of Mathematical Research & Exposition
Vol.12, No.4, Nov. 1992

Random Metric Spaces and Their Applications

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1. Introduction

It is well known that a set s of all random vatiables of a probability space (Ω, σ, μ) into a separable metric space (M, d) can form an E-space[1], because for any $p, g \in s, d(p, g)$ is a random variable. But if (M, d) is a nonseparable metric space, maybe $d(p, g)$ isn't a random variable. We haven't see any paper about the spaces of the random variables with value in a nonseparable metric space up to now.

In this paper, first, we show that a set of random variables of a probability space into a metric space (separable or nonseparable) can form a random metric space, and a set of all random variables of a probability space into a normed linear space can be embeded into a randim normed linear space. Then we indicate the application in random operators. Finally, the representation of the almost surely bounded random linear functionals is given unifiedly.

2. Preliminaries

Throughout this paper (Ω, σ, μ) denotes a probability measure space, $L^+(\Omega)$ the set of almost surely finite and nonnegative real- valued random variables, J the set of all positive integers, D^+ the set of all nondecreasing left-continuous functions $F : R \to [0,1]$ with $F(0) = 0, \inf\limits_x F(x) = 0$, and $\sup\limits_x F(x) = 1$.

For detailed definitions of probabilistic metric space (briefly, PM space), random metric space (briefly, RM space), probabilistic normed space (briefly, PN space), we refer to [1].

Definition 2.1 *A random normed space (briefly, RN space) is an ordered pair (S, X), where S is a linear space over number field k, and X is a mapping from S into $L^+(\Omega)$ such that for any $p, g \in s, \alpha \in k$:*

(1) $Xp(\omega) = 0 \quad a.s. \ \Leftrightarrow \rho = 0,$

(2) $X\alpha p(\omega) = |\alpha| \cdot Xp(\omega) \quad a.s.,$

(3) $Xp + q(\omega) \leq Xp(\omega) + Xq(\omega) \quad a.s..$

Definition 2.2 *A mapping V of a probability space (Ω, σ, μ) into a metric space (M, d) is called:*

(1) a M-valued random variable if the inverse image of every Borel subset of set M under the mapping V belings to σ.

(2) a simple random variable if there exists a finite subdivision $\{A_n\}_{n=1}^k \subset \sigma$ satisfying $\cup_{n=1}^k A_n = \Omega$ and $\{X_n\}_{n=1}^k \subset M$ such that $V(\omega) = X_n$ for $\omega \in A_n$, $n = 1, 2, \cdots, k$.

(3) a strong random variable if there exists a sequence of the simple random variables $\{V_n\}$ such that $\{V_n\}$ almost surely converges to V.

Definition 2.3 *Let $(S_i, \mathcal{F}^{(i)})$ $(i = 1, 2)$ be PM spaces. Then $(S_1, \mathcal{F}^{(1)})$ is isometric to $(S_2, \mathcal{F}^{(2)})$, if there exist a bijective mapping $\psi : S_1 \to S_2$ such that $\mathcal{F}^{(2)}(\psi p, \psi q) = \mathcal{F}^{(1)}(p, q)$ for any $p, q \in S_1$. Meanwhile $(S_1, \mathcal{F}^{(1)})$ is isometrically isomorphism to $(S_2, \mathcal{F}^{(2)})$, if $(S_i, \mathcal{F}^{(i)})$ are PN spaces and ψ is a linear mapping.*

Definition 2.4 *Let $(B, \|\cdot\|)$ be a normed linear space over number field k, B^* be its conjugate space. Then*

(1) $V : \Omega \to B^*$ is called a weak random variable, if for any $f \in B^*$, $f \circ V$ is a k-valued random variable.

(2) $V : \Omega \to B^*$ is called a W^*-random variable, if for any $x \in B$, $\langle V(\omega), x\rangle$ is a k-valued random variable.

Definition 2.5 *Let E be the set of all almost surely finite real-valued random variables defined on (Ω, σ, μ). For any $f, g \in E$, $f \le g$ iff $f(\omega) \le g(\omega)$ a.s. . Then $(E, \le, \vee, \wedge)$ is a conditional complete lattice. Let $A \subset E$, and $r : \Omega \to [-\infty, +\infty]$ be a almost surely finite real-valued function (may be not a random variable). r is called a generalized lower bound of A, if $a \ge r$ for any $a \in A$. The generalized upper bound of A can be defined analogously.*

Denote that $(E, \mathcal{F})$ is a complete PM space, endowed with probabilistic metric $\mathcal{F} : E \times E \to D^*$ defined by $\mathcal{F}(p, q) = F_{pq} = \mu\{\omega \in \Omega \mid |p(\omega) - q(\omega)| < t\}$. Further discussing, we have that $(E, \mathcal{F}, \le)$is a topological lattice.

Lemma 2.1 *Let A be a subset of E with a generalized lower (upper) bound r, lower orientable (upper orientable)*

(1) A has the infimum $\varsigma = \wedge A$ (supremum $\eta = \vee A$);

(2) there exists a decreasing sequence$\{a_n\}$ (a increasing sequence $\{b_n\}$) in A such that $\{a_n\}$ almost surely conveges to ς($\{b_n\}$ a.s. conveges to η);

(3) $\varsigma \ge r(\eta \le r)$.

In addition, if A is closed then ς is the smallest element of $A(\eta$ is the greatest element of $A)$, and we have $\varsigma = \wedge A \in A(\eta = V A \in A)$.

3. Main results

Theorem 3.1 *Suppose S is a set of random variables of a probability space (Ω, σ, μ) into a metric space (M, d). Then there exists a mapping $X : S \times S \to L^+(\Omega)$, such that (S, X) is a random metric space, and the mapping X satisfies the following:*

(1) $d(p(\omega), q(\omega)) \leq Xpq(\omega)$, for any $p, q \in s$, where $Xpq(\omega) = X(p,q)(\omega)$.

(2) $Xpq(\omega) = d(p(\omega), q(\omega))$, when $d(p,q)$ belongs to $L^+(\Omega)$;

(3) if the elements in S are strong random variables, then the $(S, \mathcal{F})$ is an E-space with base (Ω, σ, μ) and target (M, d).

Proof Fof any $p, q \in S$, let

$$A_{pq} = \{\varsigma \in L^+(\Omega) \mid \varsigma(\omega) \geq d(p(\omega), q(\omega)), \text{ a.s. }\}.$$

We first prove that A_{pq} is nonempty. In fact, for any fixed $x_0 \in M$, it is easy to prove $d(p(\omega), x_0), d(x_0, q(\omega)) \in L^+(\Omega)$ (see (2)).

Since $d(p(\omega), q(\omega)) \leq d(p(\omega), x_0) + d(x_0, q(\omega))$, we have $d(p(\omega), x_0) + d(x_0, q(\omega)) \in A_{pq}$. Hence A_{pq} is nonempty.

Let $r = d(p(\omega), q(\omega)).r$ is a generalized lower bound of A. Since A_{pq} is a closed, lower orientable subset of $L^+(\omega)$, so there exists a random variable $\varsigma \in A_{pq} \subset L^+(\Omega)$ such that ς is the smallest of A_{pq}, i.e., $\varsigma = \wedge A_{pq}$. Clearly, $\varsigma \geq d(p,q)$.

Difine $X : S \times S \to L^+(\Omega)$ by $X_{pq} = X(p,q) = \wedge A_{pq}$, so that $X_{pq} \in A_{pq} \subset L^+(\Omega)$, and $X_{pq} \geq d(p,q)$.

Obviously, if $d(p,q) \in L^+(\Omega)$, then $d(p,q) = \wedge A_{pq} = X_{pq}$.

In order to prove that (S, X) is a random metric space, we only need to show the triangular inequlity (see [1]).

Since $d(p(\omega), r(\omega)) \leq d(p(\omega), q(\omega)) + d(q(\omega), r(\omega))$, for any $p, q, r \in S$, and every $\omega \in \Omega$,

$$d(p(\omega), q(\omega)) \leq X_{pq}(\omega) \quad a.e.; \quad d(q(\omega), r(\omega)) \leq X_{pq}(\omega) \quad a.e.,$$

it is clear that $d(p(\omega), r(\omega)) \leq X_{pq}(\omega) + X_{qr}(\omega)$ a.e. , and $X_{pq} + X_{qr} \in A_{pr}$. But $X_{pr} = \wedge A_{pr}$, hance $X_{pr} \leq X_{pq} + X_{qr}$.

For the proof of (3), refer to [1].

Theorem 3.2 *Let L be a set of random variable of (Ω, σ, μ) into a normed linear space $(B, \|\cdot\|)$ over number field k, S be a linear space spanned by L (i.e.,every element of S a linear combination of elements of L). There exists a mapping $X : S \to L^+(\Omega)$ such that (S, X) is a random normed space and for any $P \in S$:*

(1) $\|p(\omega)\| \leq Xp(\omega)$, for any $p \in S$, where $X_p(\omega) = X(p)(\omega)$;

(2) $X_p(\omega) = \|p(\omega)\|$, when $\|p(\omega)\| \in L^+(\Omega)$;

(3) *if the elements of L are strong random variables, the (S,X) is an E-norm space ([5]) with base (Ω,σ,μ) and target $(B,\|\cdot\|)$.*

Proof is similar to Theorem 3.1.

Theorem 3.3 *Let S be a linear space consisting of W^*-random variables of (Ω,σ,μ) into the conjugate space B^* for a normed linear space $(B,\|\cdot\|)$. Then there exists a mapping $X: S \to L^+(\Omega)$ such that (S,X) is a random normed space, and satisfies:*

$$X_p(\omega) = X(p)(\omega) = 0 \text{ a.s. } \Leftrightarrow \langle p(\omega), x\rangle = 0 \text{ a.s. }, \text{ for any } x \in B.$$

Proof Since $|\langle p(\omega), x\rangle| \leq \|p(\omega)\|$, for we any $x \in B$ with $\|x\| \leq 1$, suppose $X_p(\omega) = \vee_{\|x\|\leq 1}|\langle p(\omega), x\rangle|, r(\omega) = \|p(\omega)\|$, and $A_p = \{|\langle p(\omega), x\rangle| \mid \|x\| \leq 1\}$.
Remanent proof is similar to theorem 3.1.

4. Applications

Definition 4.1 *Let $(B,\|\cdot\|)$ and $(Y,\|\cdot\|)$ be two normed spaces, a mapping $T: (\Omega,\sigma,\mu)\times(B,\|\cdot\|) \to (Y,\|\cdot\|)$ is called:*

(1) *a random linear operator if $T(\omega,\cdot): B \to Y$ is a linear operator for every $\omega \in \Omega$ and $T(\cdot,x): \Omega \to Y$ is y-valued random variable for all $x \in B$;*

(2) *a almost surely continuous random linear operator if there is a measurablr subset Ω_0 of Ω with $\mu(\Omega_0) = 1$ such that $T(\omega,\cdot): B \to Y$ is a continuous linear operator for every $\omega \in \Omega_0$;*

(3) *a continuous random linear operator in probability if for any $x_n \to x_0$ and every $\varepsilon > 0$*

$$\mu\{\omega \in \Omega \mid \|T(\omega, x_n - x_0)\| > \varepsilon\} \to 0, \ n \to \infty;$$

(4) *a almost surely bounded random linear operator if there exists $C(\omega) \in L^+(\Omega)$ such that*

$$\|T(\omega,X)\| \leq C(\omega)\|X\| \quad a.s. \ .$$

Remark If Y is nonseparable, then the sum of two Y-valued variable may not be a random variable. Hence the sum of random operators may not be a random operator. But we consider the random normed space (S,X) spanned by Y-valued random variables. A random linear operator can be regarded as a linear operator from $(B,\|\cdot\|)$ into (S,X). So two random linear operator, regarded as linear operators from $(B,\|\cdot\|)$ into (S,X), can be made algebraic operations.

Theorem 4.1 *Let (S,X) be a complete random metric space, $T: S \to S$ be an almost surely contraction mapping, i.e., there exists $\alpha(\omega) \in L^+(\Omega)$ with $0 \leq \alpha(\omega) < 1$ a.s. , such that $X_{TpTq}(\omega) \leq \alpha(\omega)X_{pq}(\omega)$ $a\cdot s$, for any $p,q \in S$, then T has the only fixed point.*

Remark Using Theorem 3.1, we have that Theorem 4.1 is far wider than the conclusions in [2].

Theorem 4.2 *Let $\{T_\alpha \mid \alpha \in \Delta\}$ be a random linear operator family of $\Omega \times (X, \|\cdot\|)$ into $(Y, \|\cdot\|)$, which are continuous in probability, where Δ is a index set $(B, \|\cdot\|)$ a Banach space if $\{T_\alpha \mid \alpha \in \Delta\}$ is pointwise bounded, i.e., for every $x \in X$.*

$\{T_\alpha(\cdot, X) \mid \alpha \in \wedge\}$ is a probabilistically bounded set, then $\{T_\alpha \mid \alpha \in \Delta\}$ is a equicontinuous random linear operator family.

Using Corollary 3 in [6], we can complete the proof.

Lemma 4.1[4] *Let $M(\Omega, \sigma, \mu), L(\Omega, \sigma, \mu)$ be the set of all essential bounded K-valued random variables and bounded K-valued random variables, respectively. $x \in M, y \in L$, define $\|x\|_M = \inf\limits_{\mu(A)=0} \sup\limits_{\omega \in \Omega \setminus A} |x(\omega)|, \|y\|_L = \sup\limits_{\omega \in \Omega} |y(\omega)|$. Then $(M, \|\cdot\|_M)$ and $(L, \|\cdot\|_L)$ are Banach space. There exists a linear isometric mapping $J : M \to L$ with $Jx = x$ a.s. , for every $x \in X$. Where K is R or C.*

Theorem 4.3 *Let $(B, \|\cdot\|)$ be a normed linear space over number field $K . f : \Omega \times B \to k$ is an almost surely bounded radom linear functional (i.e., there exists $C(\omega) \in L^+(\Omega)$ such that for every $x \in B |f(x, \omega)| \le C(\omega)\|x\|$ a.s. .), if and only if there is a W^*- random variable V of Ω into B^*, such that for every $x \in B$, $f(x, \omega) = \langle V(\omega), x\rangle$* a.s..

Proof Since for every $x \in B$, $|f(x, \omega)| \le C(\omega)\|x\|$ a.s., hence

$$\frac{|f(\omega)|}{C(\omega)+1} \le \|x\| \quad \text{a.s..}$$

So

$$\frac{|f(\omega)|}{C(\omega)+1} \in M(\Omega, \sigma, \mu). \tag{$*$}$$

Using Lemma 4.1, we suppose

$$\tilde{f}(\omega, x) = (C(\omega)+1) J[\frac{f(\omega, x)}{C(\omega)+1}].$$

It follows that $\tilde{F}(\omega, x) = f(\omega, x)$ a.s..

Since $C(\omega) \in L^+(\Omega)$, there is no harm in assuming $C(\omega) < +\infty$, hence for every $\omega \in \Omega, \tilde{f}(\omega, \cdot)$ is a bounded linear functional on B. In fact, using $(*)$ and Lemma 4.1, we have

$$\begin{aligned} |\tilde{f}(\omega, x)| &\le [C(\omega)+1] \cdot \sup_{\omega \in \Omega} | J[\frac{f(\omega, x)}{C(\omega)+1}]| \\ &= [C(\omega)+1] \cdot \inf_{\mu(A)=0} \sup_{\omega \in \Omega \setminus A} |\frac{f(\omega, x)}{C(\omega)+1}| \le [C(\omega)+1] \cdot \|x\|. \end{aligned}$$

Let $V : \Omega \to B^*$ defined by $V(\omega) = \tilde{f}(\omega, \cdot)$. Then it is easily proved that $V(\omega)$ is W^*-random variable. Therefore $f(\omega, x) = \tilde{f}(\omega, x) = \langle V(\omega), x\rangle$ a.s.. Inversely, if there is a W^*-random variable $V : \Omega \to B^*$ such that $f(\omega, x) = \langle V(\omega), x\rangle$ a.s.. Let$C(\omega) = \vee_{\|x\| \le 1} |\langle V(\omega), x\rangle| \in L^+(\Omega)$, we have $|f(\omega, x)| \le C(\omega) \cdot \|x\|$ a.s., for eveay $x \in B$.

Remark The RN space consisting of all almost surely bounded linear functionals on is isometrically isomorphism to the RN space, which consists of all W^*-random variables.

Example 1 Let $f : \Omega \times C[a,b] \to R$ be a continuous random linear functional. Then there is a W^*-random variable $V : \Omega \to B[a,b]$, such that $f(\omega, x(t)) = \int_a^b x(t) dv(\omega)(t)$, where $B[a,b]$ is Banach space, which consists of all bounded variation functions.

Example 2 Let $f : \Omega \times L^p[a,b] \to R$ be a continuous random linear function with $p > 1$. Then there is a random variable $V : \Omega \to L^q[a,b]$, such that $f(\omega, x(t)) = \int_{[a,b]} x(t) v(\omega)(t) dm$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Example 3 Let $(B, \|\cdot\|)$ be a reflexive Banach space, $f : \Omega \times B \to K$ be a continuous random linear functional. Then there exists a weak random variable $V : \Omega \to B^*$ such that $f(\omega, x) = \langle V(\omega), x \rangle$.

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随机度量空间及其应用

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摘 要

首先证明取值于度量空间（可分或不可分）的随机元可构成随机度量空间；取值于赋范空间的随机元可嵌入到随机赋范空间中. 接着给出这些结论对随机算子的应用. 最后统一给出赋范空间上几乎处处有界的随机线性泛函的表示.