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Zero–Product-Associative Reduced Near-Rings*

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Abian [1] introduced an order relation $\leq$ (Abian's order relation) in a reduced ring R by defining $x \leq y, x, y \in R$, if and only if $x^2 = xy$. In [2,3], the relation $\leq$ is introduced in a (not necessarily associative) zero-product-associative (ZPA) reduced ring R and it is shown that $\leq$ is a partial order relation in R. We will study Abian's order of the ZPA reduced near-rings. Let N be a ZPA, ZPD reduced near-ring with identity $1. A$ is the set of all idempotents of N. We show that $(A, \leq)$ is a lattice and $(A, \wedge, \vee, ', 0, 1)$ forms a Boolean algebra by defining $e \wedge f = ef, e \vee f = e + f - ef$ and $e' = 1 - e$. All results in this paper generalize the parallel results in [1–4].

An algebra system $N = (N, +, \cdot, 0)$ is called a (zero-symmetric) near-ring if (i) $(N, +, 0)$ is a (not necessarily Abelian) group, (ii) $(N, \cdot)$ is a semigroup, (iii) $x(y + z) = xy + xz$, for all $x, y \in N$, and (iv) $0x = 0$, for all $x \in N$. A (not necessarily associative) zero-symmetric near-ring N is said to be zero-product-associative (ZPA) if a product of elements of N which is equal to zero remains equal to zero no matter how its factors are associative. A ZPA near-ring N without nonzero nilpotent elements is called a ZPA reduced near-ring. Other terminologies can be found in [5].

We check easily that the following results in [2] is true for ZPA near-ring by similar arguments.

Lemma 1 *A ZPA near-ring N is reduced if and only if $x^2 = 0$ implies $x = 0$, for all $x \in N$.*

Lemma 2 *In a ZPA reduced near-ring N, $xy = 0$ implies $yx = 0$, for all $x, y \in N$.*

Lemma 3 *A ZPA reduced near-ring N has insertion-of- factors property, that is, $xy = 0, x, y \in N$, implies $xny = 0$, for any $n \in N$.*

Lemma 4 *In a ZPA reduced near-ring $N, x_1 \cdots x_m = 0$ (elements x's not necessarily distinct) if and only if $y_1 \cdots y_n = 0$, where $y_1 \cdots y_n$ is a product (in any order whatsoever) of all the distinct factors appearing in $x_1 \cdots x_m$.*

Now we introduce the notion of zero-product-distributive. A ZPA near- ring N is said to be zero-product-distributive (ZPD) if and only if $xy = 0, x, y \in N$, implies $(x + y)n = xn + yn$, for all $n \in N$. Many algebra systems are ZPA ZPD near-rings. For example, ZPA rings, integral near-ring, and a reduced near-rings is ZPD since we have the following

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proposition.

Proposition 5 *Every reduced near-ring N is ZPD.*

Proof Let $x, y \in N$ such that $xy = 0$, then we have $yx = 0$. Hence we have $x((x+y)d - yd - xd) = 0$ and $y((x+y)d - yd - xd) = 0$, so $((x+y)d - yd - xd)x = 0$ and $((x+y)d - yd - xd)y = 0$. Hence $((x+y)d - yd - xd)(x+y) = 0$, so $((x+y)d - yd - xd)^2 = 0$. Since N is reduced, we have $(x+y)d - td - xd = 0, (x+y)d = xd + yd$.

Lemma 6 *If $x(y-x) = 0, x, y \in N$, where N is ZPD, then $(y-x)n = yn - xn$, for all $n \in N$.*

Proof $x(y-x) = 0$ implies $(y-x)x = 0$ by Lemma 2. N is ZPD, so $yn = ((y-x)+x)n = (y-x)n + xn$, hence $(y-x)n = yn - xn$.

We introduce Abian's order relation $\leq$on a ZPA ZPD near ring by defining $x \leq y$ if and only if $x^2 = xy$. Since we have Lemma 6, the proof of the following theorem is the translation of the proof of the similar theorems in [2,3] and so will be omitted.

Theorem 7 *Let N be ZPA ZPD near-ring. Then Abian's order relation $\leq$ is a partial order on N if and only if N is reduced. In this case, $(N, \cdot, \leq)$ is a partially ordered groupoid, namely if $x \leq y$ and $a \leq b$, then $xa \leq yb$.*

Theorem 8 *If X is a subset of N such that $\sup X$ exists, then for $r \in N$, $\sup rX$ and $\sup Xr$ exist, and (i) $\sup rX = r \sup X$, (ii) $\sup Xr = (\sup X)r$.*

From now on, let N be a ZPA ZPD reduced near-ring with identity 1 and $\leq$ Abian's partial order.

Lemma 9 *Let S be a subset of idempotent elements of N, if $\sup S(\inf S)$ exists, then it must be idempotent.*

Proof Denote $\sup S = x$. Then $s \leq x$, for all $s \in S$, so $s = s^2 = sx$ or $s(x-1) = 0$. Note that N is ZPA, from Lemma 3, we have $0 = s(x(x-1)) = s(x^2 - x)$ or $sx^2 = sx = s^2$, so $s \leq x^2$, for all $s \in S$, i.e., x^2 is an upper bound of S. Hence $x \leq x^2$, i.e., $x^2 = xx^2$ or $x(x - x^2) = 0$. By Lemma 2 we have $(x - x^2)x = 0$, so $(x - x^2)x^2 = 0$, and so $(x - x^2)(-x^2) = 0$. From these facts, we have $(x - x^2)^2 = 0$, so $x - x^2 = 0$, i.e., $x = x^2$ since N is reduced. Hence $x = \sup S$ is idempotent.

Denote $\inf S = y$, then $y \leq s$, for all $s \in S$, i.e., $y^2 = ys$ or $y(y-s) = 0$. From Lemma 3, we have

$$0 = y(s(y-s)) = y(sy - s^2) = y(sy - s) = (ys)(y-1) = y^2(y-1) = y(y(y-1)) = y(y^2 - y),$$

so $(y^2 - y)y = 0$, and so $(y^2 - y)y^2 = 0$ by Lemma 4. Consequently $(y^2 - y)^2 = 0$, so $y^2 - y = 0, y^2 = y$, i.e., $\inf S = y$ is idempotent.

Lemma 10 *Let e be an idempotent element of N. Then*
(i) $e(ne) = ne$, for each $n \in N$, (ii) e is in the center of N.

Proof (i) From $0 = e(1-e)$ we have $(1-e)e = 0$ and $0 = (1-e)(ne)$ for each $n \in N$. Thus $ne = (1 - e + e)(ne) = (1-e)(ne) + e(ne) = e(ne)$.

(ii) Since $e(1-e)=0$, $0=e(n(1-e))=e(n-ne)=en-e(ne)$, we have $en=e(ne)$. From this and (i) we get $ne=en$, for each $n\in N$.

Lemma 11 *Let e be an idempotent element of N. Then $en+n=n+en$ and $ne+n=n+ne$, for each $n\in N$.*

Proof It is enough to show $en+n=n+en$. From $e(e+1-e-1)=0$, we get $(e+1-e-1)e=0$. From this we have $(e+1-e-1)^2=0$, so $e+1-e-1=0$, i.e., $e+1=1+e$. Since $en+n=ne+n=n(e+1)$ and $n+en=n+ne=n(1+e)$, we get $en+n=n+en$.

Theorem 12 *Let e and f be idempotent elements of N, $\leq$ Abian's order. Then*
(i) $ef=\inf\{e,f\}$, *(ii)* $e+f-fe=\sup\{e,f\}$.

Proof (i) As $0=e(f-f)=e(f^2-f)=(ef)(f-1)=(fe)(e(f-1))=(fe)(ef-e)=(fe)(fe-e)$, so $(fe)^2=(fe)e$, i.e., $fe\leq e$. Similarly we can show that $fe\leq f$. From these facts, we have that fe is a lower bound of $\{e,f\}$.

Suppose x is any lower bound of $\{e,f\}$, then $x\leq e$ and $x\leq f$, i.e., $x^2=xe=xf$, so

$$\begin{aligned} 0 &= x(f-f)=x(f-f^2)=(xf)(1-f)=fx(1-f) \\ &= f(x(1-f))=f(x-xf)=f(x-xe) \\ &= fx(1-e)=xf(1-e)=x(f-fe)=xf-x(fe)=x^2-x(fe), \end{aligned}$$

therefore $x^2=x(fe)$, i.e., $x\leq fe$.

From above facts, we have $fe=\inf\{e,f\}$.

(ii) Since $e(e-(e+f-fe))=e(e+fe-f-e)=e+e(fe)-ef-e=e+fe-fe-e=0$ by Lemma 10. Hence $e^2=e(e+f-fe)$, i.e., $e\leq e+f-fe$.

Similarly, $f(f-(e+f-fe))=f(f+fe-f-e)=f+fe-f-fe=fe+f-f-fe=0$ by Lemma 10 and Lemma 11. Hence $f^2=f(e+f-fe)$, i.e., $f\leq e+f-fe$. So $e+f-fe$ is a upper bound of $\{e,f\}$.

Let y be any upper bound of $\{e,f\}$. Then $e\leq y$ and $f\leq y$, i.e., $e=e^2=ey$ and $f=f^2=fy$. From $f(f-1)=0$ implies $0=f(e(f-1))=f(ef-e)=f(fe-e)=f(e(fe)-e)=f(e(fe-1))=(fe)(fe-1)=(fe)^2-fe$, so $(fe)^2=fe$.

Since $e(f-fe)=0$ and $(fe)(f-fe)=0$, by Lemma 6 we have

$$\begin{aligned} (e+f-fe)(fe) &= (e+(f-fe))(fe)=e(fe)+(f-fe)(fe)=fe+f(fe)-(fe) \\ &= fe+fe-fe=fe. \end{aligned}$$

So we get

$$(e+f-fe)(fe)=fe. \tag{1}$$

We also have

$$(e+f-fe)y=ey+fy-(fe)y=e+f-(fe)y$$

and $0=f(e-e)=f(ey-e)=f(e(y-1))=(fe)(y-1)=(fe)y-fe$, i.e., $(fe)y=fe$, therefore we obtain

$$(e+f-fe)y=e+f-fe. \tag{2}$$

From (1), (2) and Lemma 11 we obtain

$$\begin{aligned}
& (e+f-fe)((e+f-fe)-y) \\
= \;& (e+f-fe)e+(e+f-fe)f-(e+f-fe)(fe)-(e+f-fe)y \\
= \;& e(e+f-fe)+f(e+f-fe)-fe-(e+f-fe) \\
= \;& e+fe-fe+(fe+f)-fe-fe+fe-f-e \\
= \;& e+(fe+f)-fe-f-e=e+f+fe-fe-f-e=0.
\end{aligned}$$

Hence $e+f-fe\leq y$, so $e+f-fe$ is the suprema of $\{e,f\}$, namely, $\sup\{e,f\}=e+f-fe$.

The partial order relation $\leq$ in N induces a partial order relation which we also denote by $\leq$ in A, the set of all idempotent elements of N. From Theorem 12 we obtain easily the main results of this paper.

Theorem 13 *$(A,\leq)$ is a lattice with 0 and 1.*

Theorem 14 *We define $\wedge,\vee$ and $'$ on A as follows. Let $e,f\in A$. Define $e\wedge f=ef, e\vee f=e+f-fe$ and $e'=1-e$. Then $(A,\wedge,\vee,',0,1)$ is a Boolean algebra.*

Theorem 15 *$(A,\cdot)$ forms a semigroup.*

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零积结合约化近环

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摘　要

本文引入零积可分配近环的概念，研究零结合零积可分配约化近环 N 中的 Abian 序$\leq$，我们的主要结果是证明了，如果 N 具有恒等元 1，则 N 的全体幂等元之集 A 对于 Abian 序$\leq$成为一个格；在 A 中定义 $e\wedge f=ef$, $e\vee f=e+f-fe$, $e'=1-e$, $(A,\wedge,\vee,',0,1)$ 作成一个布尔代数；而且虽然 $(N,\cdot)$ 是非结合的，$(A,\cdot)$ 却成为一个半群.