Journal of Mathematical Research & Exposition
Vol.12, No.4, Nov. 1992

# Structure and Application of Algebraic Spline Curve and Surface*

*Luo Zhongxuan* *Wang Renhong*
(Inst. Math. Sciences, Dalian Univ. of Tech., Dalian)

**Abstract** Algebraic-geometry foundations for non- linear spline functions are established. The structure of algebraic spline curves and surfaces represented by implicit forms are investigated. The necessary and sufficient conditions for smooth connection of algebraic curves(or surfaces) are presented.

## 1.Definitions and Notations

An algebraic curve of degree of $n$ is determined by a certain bivariate polynomial equation

$$a_{00}+a_{10}x+a_{01}y+\cdot+a_{0n}y^n=0,$$

where $a_{ij}\in K$, $\sum_{i+j=n}a_{ij}^2\neq 0$, and $K$ is a field.

**Notations** *Denote by $K[x]'$ the field of formal power series over $K$ in $x$,and by $K(x)^*$ the field of extended fractional power series over $K$ in $x$.*

**Definition** *The definitions of Newton polygon and characteristic equation of an algebraic curve are referred to R.J.Walker*[1]. *For a curve $f(x,y)=0$ , we take a transformation*

$$y=x^{r_1}(c+y),$$

*where $-r_1$ is the gradient of the boundary segment in the Newton polygon previously considered and c is a non-zero root of the Newton characteristic equation. Then we can get another equation of $f_1(x,y)=0$. The Newton polygon and characteristic equation of $f_1(x,y)=0$ are called the second Newton characteristics, etc.,so we can define r-th Newton characteristics.*

The following conclusions will be used.

**Theorem1.1**(Puiseux's theorem) *The field $K(x)^*$ is algebraically closed.*

**Lemma 1.1** *There is a unique set of elements $\bar{y}_1,\cdots,\bar{y}_n\in K(x)^*$ such that $f(x,y)=\bar{a}_n\prod_{i=1}^n(y-\bar{y}_i)$.*

**Lemma 1.2** *If $O(\bar{a}_n)\le O(\bar{a}_i), i=0,1,\cdots,n-1$, then all the elements $\bar{y}_i$ in Lemma 1.1*

*Received Oct. 6, 1990.

*belong to $K[x]^*$. $K[x]^*$ is a domain whose elements are in $K(x)^*$ of non-negative order.*

**Lemma 1.3** *Soppose $O(\overline{a}_n) = 0, O(\overline{a}_0) > 0$, $O(\overline{a}_i) \geq 0, i = 1, 2, \cdots, n-1$,, then for at least one of the $\overline{y}_j$ of Lemma 1.1, $O(\overline{y}_j) > 0$.*

## 2.Structural Characteristics of Algebraic Spline Curves

In this section, we shall consider the problem of connection between algebraic curves.

Let curves of degree of $n$ and $m$ be determined by $C_1 : f_1(x,y) = 0$ and $C_2 : f_2(x,y) = 0$, respectively.

The following conclusions are true.

**Theorem 2.1** *Algebraic curves $C_1$ and $C_2$ are connected at origin iff one of the following conditions holds:*

*1. $K > -1$(or $K < -1$), where $K$ is the gradient of a common boundary segment of Newton polygons of curves $C_1$ and $C_2$;*

*2.the two Newton characteristic equations corresponding to a common boundary segment of $C_1$ and $C_2$ have a common real root, when the gradient of the common segment is $-1$.*

**Remark** *If $K$ is a complex field and if the power series $\overline{a}_i(x)$ converges in a domain $<$, then each root of $f(x,y) = 0$ can be shown to be convergent in similar domain.*

Let $s(\mu) = max\{s : r_1 + \cdots + r_s \leq \mu\}$, where $-r_i(i = 1, 2, \cdots, s)$ are the gradient of the boundary segment in $i$-th Newton polygon of an algebraic curve $f(x,y) = 0$.

Generally, we have

**Theorem 2.2** *Algebraic curves $C_1$ and $C_2$ are $C^r$-connected iff one of the following conditions holds:*

*1. there are $i$-th($i < s(r)$) Newton polygon of $C_1$ and $C_2$ such that the gradient of their common boundary segment is greater than $-1$;*

*2.there always exists a common boundary segment of $i$-th($i = 1, 2, \cdots, s(r)$) Newton polygon of $C_1$ and $C_2$ , which has the same gradient $-k$($k \geq 1$,is an integer), and Newton characteristic equations corresponding to the equations have a common root.*

From Theorem 2.2, we can easily obtain

**Corollary 2.1** *If curves $C_1 : g(x,y) = 0$ and $C_2 : h(x,y) = 0$ have no common component, and if the gradient of a boundary segment of Newton polygon of $C_2$ is $-r$. Then curve $f(x,y) = g(x,y) + h(x,y) = 0$ and $g(x,y) = 0$ are $C^{r-1}$-connected at the intersection point of $C_1$ and $C_2$.*

The following results are extensions of Theorem 2.2.

**Proposition 2.1** *For the situation conic, there are algebraic spline curves(non-degenerate) of $C^2$ and $C^3$.*

**Proposition 2.2** *Two irreducible conics $C_1$ and $C_2$ are $C^2$ connective at the intersection point of $C_1$ and $C_2$ iff there is a linear function $P_1(x,y)$ such that*

$$f_1(x,y) - f_2(x,y) = l(x,y)P_1(x,y),$$

*where $l(x,y)=0$ is the tangent equation of $C_1$ and $C_2$ at the intersection point $(x^*,y^*)$ and $P(x^*,y^*)=0$.*

**Proposition 2.3** *Two irreducible conics $C_1$ and $C_2$ are $C^3$ connective at the intersection point of $C_1$ and $C_2$ iff there is a constant $C$ such that*

$$f_1(x,y)-f_2(x,y)=C[l(x,y)]^2,$$

*where $l(x,y)=0$ is the tangent equation of $C_1$ and $C_2$ at the intersection point.*

Generally, we can get the following crucial result.

**Theorem 2.3** *Two algebraic curves of degree $n$ are $C^{n+k}(0\le k\le n-1)$-connected iff there exists a polynomial $Q(x,y)\in \mathbf{P}^2_{n-(k+1)}$ such that*

$$f_1(x,y)-f_2(x,y)=Q(x,y)[l(x,y)]^{k+1},$$

*wherel$(x,y)=0$ is the tangent equation of $C_1$ and $C_2$ at the intersection point.*

**Example** Construct an algebraic spline curve, such that the curve cosses the points $A_1(0,0),A_2(1,1),A_3(2,1.5),A_4(3,2.5)$ and satisfies $y^{(1)}(0)=3, y^{(2)}(0)=-20, y^{(3)}(0)=360$, with $C^3$-smoothness on $[0,3]$.

$$\begin{aligned}
&C_1: -3x+x^2+y+y^2=0, \qquad x\in[0,1]\\
&C_2: 28+31x-85y+6x^2-42xy+62y^2=0, \quad x\in[1,2]\\
&C_3: 8(x-2)+\frac{29818}{1377}(x-2)^2-17(y-\frac{3}{2})\\
&+(\frac{561}{34}-\frac{289}{68}\frac{29818}{1377}(x-2)(y-\frac{3}{2})\\
&+(\frac{289}{64}\frac{29818}{1377}-\frac{1117}{32})(y-\frac{3}{2})^2=0, x\in[2,3].
\end{aligned}$$

In view of the fact that the conic has no flex point, it can be verified directly that if the nodes of spline curve determine a convex polygon, then the corresponding conic spline curve is also convex.

## 3. Structure Characteristics of Algebraic Spline Surfaces

We define an infinite sum of the following type:

$$\sum_{i+j\ge 0} a_{ij}x^iy^j,\, a_{ij}\in K,$$

where $K$ is a field. So we obtain bivariate formal series over $K$ of $K[x,y]'$.

If all elements of $K[x,y]'$ are convergent, it is known that $K[x,y]'$ is a unique factorization domain, and $K(x,y)=K[x^{1/m},y^{1/n}]'$ is a domain.

Let the equation of an algebraic surface can be written in form

$$\begin{aligned}
f(x,y,z) &= \sum_{i=0}^{n} a_i^{(0)}(y)x^i+(\sum_{i=0}^{n-1} a_i^{(1)}(y)x^i)z+\cdots+a_nz^n\\
&= f_n^{(y)}(x)+f_{n-1}^{(y)}(x)z+\cdots+f_0^{(y)}z^n, \quad (3.1)
\end{aligned}$$

which can be expressed symmetrically as

$$\begin{aligned} f(x,y,z) &= \sum_{i=0}^{n} a_0^{(i)}(x)y^i + (\sum_{i=0}^{n-1} a_1^{(i)}(x)y^i)z + \cdots + a_n z^n \\ &= g_n^{(x)}(y) + g_{n-1}^{(x)}(y)z + \cdots + g_0^{(x)} z^n, \quad (3.2) \end{aligned}$$

Suppose that

$$A: \qquad O(f_0^{(y)}(x)) \le O(f_i^{(y)}(x)), O(g_0^{(x)}(y)) \le O(g_i^{(x)}(y)), \quad i = 1, 2, \cdots, n,$$

and that the Newton polygons of (3.1) and (3.2) on $x$ and $y$ are identical, respectively.

Denote by $z \to x \to y$ the process of solving the power series solutions of $f(x,y,z) = 0$ for $x$ under the assumption above.

**Theorem 3.1** *The processes $z \to x \to y$ and $z \to y \to x$ are equivalent.*

Here, we get the result which is weaker than the situation of single variate.

**Theorem 3.2** *Under the assumption as above, the algebraic surface $f(x,y,z) = 0$ has at least one solution in $K(x,y)$.*

Let $\overline{AS}^n$ = {algebraic surfaces which satisfy the condition $A$}.

**Lemma 3.1** *If the algebraic curves $C_1$ and $C_2$ have the same places of curve, then $C_1$ and $C_2$ will express the same curve.*

**Theorem 3.3** *Let both surfaces $M_1$ and $M_2$ be in $\overline{AS}^n$, then $M_1$ and $M_2$ are $C^r$-connected on the line $x = 0$ iff the i- th($i = 1, 2, \cdots, r$) Newton characteristic equations on $x$ of $M_1$ and $M_2$ have the same roots.*

**Theorem 3.4** *Let $M_1 : f_1(x,y,z) = 0 \in \overline{AS}^n, M_2 : f_2(x,y,z) = 0 \in \overline{AS}^n$ and $M_2$ cross the y-axis, then algebraic surfaces*

$$M_3 : M_1 + \mu M_2^r = 0, \quad \mu \in (1,0)$$

*and $m_1$ are $C^{r-1}$-connected on $x = 0$.*

## References


[1] R. J. Walker, *Algebraic Curves,* Princeton Univ. Press, Princeton, N. J. 1950.

[2] R. H. Wang, *The structural characterization and interpolation for multivariate splines,* Acta Math. Sinica, 18(1975), 91-106.

[3] B. Q. Shu, *Computational Geometry,* Shanghai Science and Tech. Press, Shanghai, 1981.

[4] L. L. Schumaker, *Spline Function: Basic Theory,* Wiley, New York, 1981.

[5] E. L. Wachspress, *A Rational Basis for Function Approximation II: curved sides,* J. Inst. Maths. Applics. 11(1973), 83-103.