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On Star-Incidence Algebra and Star-Form Möbius Inversion*

L. C. Hsu
(Dalian Univ. of Tech., Dalian)
Sun Guangrun
(Qufu Normal Univ., Qufu, Shandong)

Abstract In the recent papers [1], [2], we showed that the Möbius inversion can be generalized to the locally infinite, point-representable poset. The point of the present paper is to exploit $*-$finite structures in the study of combinatorics. It is noted that there exists some locally $*-$finite poset C containing all the standard entities in the natural extension $^*S$ of a locally infinite poset S, and then a $*-$incidence algebra $^*I(C,{}^*K)$ of C, over a field $^*K$ of characteristic 0, is defined. It follows from this that the Möbius inversion can be generalized to general locally $*-$finite posets.

An application to some linearly ordered set is given anew of such result.

§0. Introduction

The study of the first author on the generalized Möbius inversion was begun in the articles [1], [2].

Most standard discrete structures are often endowed with some natural order structures. Thus G.-C. Rota [3] expanded the idea of L. Weisner and P. Hall, which the Möbius function and Möbius inversion were defined for functions over locally finite poset S_f. Their works are as follows. Let $f, g \in$ Map(S_f, K), the functions from S_f into a field K of characteristic 0 (usually the standard real numbers). Suppose that the locally finite poset S_f contains a greatest lower bound, $0-$element, denoted by 0 (or all principal ideals of S_f are finite). We may define the lower sum operator $S_\leq$ on Map(S_f, K) by

$$(S_\leq f)(x) = \sum_{y\leq x} f(y) \qquad (0-1)$$

and the lower difference operator $D_\leq$, inverse to $S_\leq$, by

$$(D_\leq g)(x) = \sum_{y\leq x} g(y)\mu(y,x) \qquad (0-2)$$

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analogously to the indefinite integral in calculus, where μ is the Möbius function. Then

$$g(x) = (S_{\leq} f)(x) \Leftrightarrow f(x) = (D_{\leq} g)(x), x \in S_f. \qquad (0-3)$$

In virtue of the assumption that the poset S is locally finite, the Dirichlet products of (0–1) and (0–2) are well-defined.

Is it possible to generalize (0–3) to locally infinite poset S, of which every segment is standard infinite? This is indeed so if S is point-representable poset as in [1], [2]. In the present paper, we study general locally infinite poset, for example, the power set $\mathcal{P}(M)$ of an infinite set M, which is partially ordered by set inclusion $\subseteq$ but locally infinite poset with 0−element, empty set $\emptyset$. We will show that if $S \subseteq B \subseteq M$, for any $f, g \in^* \mathrm{Map}(\mathcal{P}(M), K)$, Möbius inversion

$$g(B) = \sum_{S \subseteq B} f(S) \Leftrightarrow f(B) = \sum_{S \subseteq B} \mu(S, B) g(S) \qquad (0-4)$$

holds in some corresponding form of nonstandard combinatorics (see Corollary 3.2).

Another example is as follows. Let $\Pi(A)$ be the lattice of partition of an infinite set A, which is the set of partitions of A, ordered by refinement $\leq$. The 0−element of $\prod(A)$ is the partition whose blocks are the one element subsets of A. We will show also that for any partition and any $f, g \in^* M(\prod(A), K)$ the Möbius inversion

$$g(X) = \sum_{\mathcal{A} \leq X} f(\mathcal{A}) \Leftrightarrow f(X) = \sum_{\mathcal{A} \leq X} \mu(\mathcal{A}, X) g(\mathcal{A}) \qquad (0-5)$$

holds in the sense of *−finiteness (cf. Corollary 3.2). These examples demonstrate that any (locally) infinite set with some additional order structure should be considered.

Section 1 contains the most elementary facts of nonstandard analysis about the present paper. Here, in view of above observations, we introduce particularly the superstructure, and an enlargement of a superstructure and the exhaustiveness as well. We construct a $*$−incidence algebra in Section 2, and generalize Möbius inversion to $*$−finite sets in Section 3, obtaining star-form Möbius inversion. Section 4 illustrates with the fundamental theorem of integral calculus that $*$−finite combinaorial techniques are particularly useful in realizing a synthesis of continuous and discrete structures.

§1. Superstructure and exhaustiveness

Let X be any nonempty set. We will work in the superstructure $V(X \cup N)$ defined as follows. The n^{th} cumulative power set of $X \cup N$ is defined recursively by

$$V_0(X \cup N) = X \cup N,$$
$$V_{n+1}(X \cup N) = V_n(X \cup N) \cup \mathcal{P}[V_n(X \cup N)].$$

The superstructure over $X \cup N$ is the set

$$V(X \cup N) = \cup_{n=0}^{\infty} V_n(X \cup N).$$

Notice that the field K of characteristic 0 and all infinite entities with the additional partically order structure, e.g., the lattice of partitions $\prod(S)$ of a set S, are in the framework of superstructure. Throughout this paper, $S = (S, \leq)$ denotes a locally infinite poset.

Definition 1.1([4]) *A family $\mathcal{E}$ of subsets of an entity $S \in V(X \cup N)$ is called exhausting if, for each finite subset $F \subseteq S$, there is an $E \in \mathcal{E}$ such that $F \subseteq E$.*

Lemma 1.2 *Let $\mathcal{E}$ be the family of all locally finite subsets of $S \in V(X \cup N)$. Then $\mathcal{E}$ is an exhausting family.*

Proof Assertion is trivial since each finite subset F of S is locally finite. □

We suppose now that an index set I and some free ultrafilter on I are chosen so that the superstructure $V[^*(X \cup N)]$ (with respect to a monomorphism $^* : V(X \cup N) \longrightarrow V[^*(X \cup N)]$ is an enlargement of $V(X \cup N)$.

We will need the following well known result of nonstandard analysis.

Proposition 1.3([4]) *If $\mathcal{E}$ is an exhausting family of subsets of $S \in V(X \cup N)$ and $V[^*(X \cup N)]$ is an enlargement, then there exists a set $C \in {}^*\varepsilon$ containing all the standard entities in $^*S$.*

Corollary 1.4 *Let $\mathcal{E}$ be as in Lemma 1.2. Then there exists a locally $*-$finite partially ordered subset $C \in^* \mathcal{E}$, which contains ${}^\sigma S = \{^*a : a \in S\}$.*

§2. $*-$Incidence algebra

Lemma 2.1 *Every maximal chain in a segment $[x, y]$ of S is refinable in $'S$, that is, there exist nonstandard entities in $^*[x, y]$, which are comparable with the standard entities in S.*

Proof Let $(x_0, x_1, \cdots, x_n)$ be the maximal chain in a segment $[x, y]$ of S, that is, $x_0 = x, x_n = y$, and $x_i + 1$ covers x_i for all i. Since the superstructure $V(^*[X \cup N])$ (with respect to monomorphism $* : V(X \cup N) \longrightarrow V[^*(X \cup N)])$ is an enlargement of $V(X \cup N)$ and $[x, y]$ is infinite, $^*[x, y]$ is also infinite and contains entities which are not standard.

Let $P(x', y')$ be the relation on $[x_i, x_j], 0 \leq i \leq j \leq n$, which holds whenever $x' \leq y', x', y' \in [x_i, x_j]$. Then relation P is concurrent on dom$P$, since for each finite set $\{x_{11}, \cdots, x_{1k}\}$ in dom$P$, we may choose $y = \max\{x_{11}, \cdots, x_{1k}\}$ so that $\langle x_{11}, y\rangle \in P, 1 \leq i \leq k$. Thus it is obtained by $V[^*(X \cup N)]$ being an enlargement that for the concurrent relation $P \in V(X \cup N)$, there is an element $b \in \text{rng}^*P$ so that $\langle \dot{x}, b\rangle \in^* P$ for all $x \in \text{dom}P = [x_i, x_j]$. For the dual relation P^*, the argument is similar.

The lemma is established immediately from these. □

Because of Corollary 1.4 and Lemma 2.1, we can define the $*-$incidence algebra $^*I(C,^* K)$ of the locally $*-$finite partially ordered subset C of $^*S \in V[^*(X \cup N)]$, over a field $^*K$ that is the natural extension of a field K of characteristic 0, where C is as in §1. The members of $^*I(C,^* K)$ are $^*K$ valued functions $^*f(x, y)$ of two variables, with x and y ranging over C and with sole restriction that $^*f(x, y) = 0$ unless $x \leq y$, where $^*f$ is the natural extension of $f \in V(X \cup N)$. The sum of two such functions, as well as multiplication by standard or nonstandard scalars, are defined as usual, and the star-Dirichlet product ${}^*f *{}^*g ={}^*h$ is defined as follows

$$ {}^*h(x,y) ={}^* \sum_{z\in[x,y]}^{*} f(x,z)^*g(z,y). \tag{2-1}$$

Notice that the right-hand side is well defined which is a $*-$finite sum by Corollary (1.4). Obviously, ${}'f *' g \in' I(C,' K)$, where we set $({}'f *' g)(x,y) = 0$ if $x \not\leq y$.

Definition 2.2 *The family ${}'I(C,' K)$ together with the operations addition, standard or nonstandard scalar multiplication and Star-Dirichlet product (star-convolution) is called the $*-$incidence algebra of a locally $*-$finite poset C, over a field ${}^*K$.*

The incidence functions ${}^*\delta, {}^*\varsigma, {}^*\lambda, {}^*\mu$ and so on of ${}^*I(C, {}^*K)$ are the natural extensions of the corresponding functions of $I(S_f, K)$ respectively, e.g., we set

(a) Kronecker function ${}^*\delta(x,y) := \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$

(b) Zeta-function ${}^*\varsigma(x,y) := \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases}$

(c) Chain-function ${}^*\lambda(x,y) :={}^*\varsigma -{}^*\delta$.

(d) Möbius function ${}^*\mu :={}^*\varsigma^{-1}$.

It is easily verified by the transfer principle that the locally $*-$finite posets possess the formal combinatorial properties of the locally finite posets. For example, the following are obvious.

Proposition 2.4 *An element ${}^*f \in{}^* I(C,{}^*K)$ is a unit if and only if ${}^*f(x,x) \neq 0$ for all $x \in C$. A unit ${}^*f$ possesses a unique two-sided inverse ${}^*f^{-1}$ which is defined by Principle of Transfinite Induction.*

§3. Star-form Möbius inversion

Theorem 3.1(star-form Möbius inversion) *Let C be a locally ${}^*-$finite poset with a $0-$element 0. Then for ${}'f,' g \in{}^*Map(C,{}^*K)$*

$$ {}^*g(x) ={}^* \sum_{0\leq y\leq x} {}^*f(y)(x\in C) \Leftrightarrow {}^*f(x) ={}^* \sum_{0\leq y\leq x} {}^*g(y){}^*\mu(y,x), (x \in C) \tag{3-1}$$

holds, where ${}^*\mu(y,x)$ is the Möbius function of ${}'I(C,' K)$.

Proof The conclusion is clear by transfer. In fact, let $\mathcal{E}$ be as in (1.2), then for each $E \in \mathcal{E}$, the following sentence formulated in terms of semi-formal notations is true in $V(X \cup N)$:

$$\begin{aligned} &(\forall E)\ [E \in \mathcal{E}] \Rightarrow (\forall f)(\forall g)\ [f, g \in \mathrm{Map}(E,K)] \Rightarrow (\exists S_{\leq})(\exists D_{\leq})[S_{\leq} \in S(\mathcal{M},\mathcal{R}) \\ &\wedge D_{\leq} \in D(\mathcal{M},\mathcal{R})] \wedge (\forall x)[x \in E] \Rightarrow [g(x) = (S_{\leq}f)(x) \Leftrightarrow f(x) = (D_{\leq}g)(x)] \end{aligned} \tag{3-2}$$

where $S(\mathcal{M},\mathcal{R})$ and $D(\mathcal{M},\mathcal{R})$ denote the families of lower sum operators and lower difference operators respectively. By transfer, the $*-$transform of (3–2)

$$
\begin{aligned}
&(\forall E)(E \in^* \mathcal{E}] \Rightarrow (\forall f)(\forall g)[f, g \in^* \mathrm{Map}(E, ^*K) \Rightarrow (\exists S_{\leq})(\exists D_{\leq}) \\
&[S_{\leq} \in^* S(^*\mathcal{M}, ^*\mathcal{R}) \wedge D_{\leq} \in^* D(^*\mathcal{M}, ^*\mathcal{R}) \wedge (\forall x)[x \in E] \\
&\Rightarrow g(x) = (S_{\leq} f)(x) \Leftrightarrow f(x) = (D_{\leq} g)(x)
\end{aligned}
\tag{3 – 3}
$$

is true in $V[^*(X \cup N)]$, that is

$$
^*g(x) = (^*S_{\leq} {^*f})(x) \Leftrightarrow {^*f}(x) = (^*D_{\leq} {^*g})(x) \tag{3 – 4}
$$

is true for all $x \in E \in^* \mathcal{E}$ (E being locally $*-$finite poset).

It was noted in Corollary 1.4 that for locally infinite poset S, there is a locally $*-$finite, partially ordered subset $C \in^* \mathcal{E}$ containing all the standard entities in $^*S$. We now restrict x to S, then (3–3) holds. Thus we obtain the following generalized Möbius inversion.

Corollary 3.2 *Let S be a llocally infinite poset with a $0-$element 0. Then for $^*f, {^*g} \in {^*\mathcal{M}}(C, {^*K})$*

$$
^*g(x) = (^*S_{\leq} {^*f})(x) \Leftrightarrow {^*f}(x) = (^*\mathcal{D}_{\leq} {^*g})(x), (x \in S). \tag{3 – 5}
$$

Proof It suffices to notice the fact that any segment $^*[0, x]$, where 0 and x are in S, has a $0-$element.

Theorem 3.3 *Let S be as above. Suppose $^*\mu_1 \in^* I(C, {^*K})$ possesses a inverse $^*\mu_2 = {^*\mu_1^{-1}}$. Then*

$$
^*g(x) = {^*\sum_{y \leq x}} {^*f}(y) {^*\mu_1}(y, x) \Leftrightarrow {^*f}(x) = {^*\sum_{y \leq x}} {^*g}(y) {^*\mu_2}(y, x), (x \in S). \tag{3 – 6}
$$

Proof For $^*f, {^*g} \in^* I(C, {^*K})$, we have

$$
^*g = {^*f} * {^*\mu_1} \Leftrightarrow {^*f} = {^*g} * {^*\mu_2}.
$$

Now we define

$$
^*\bar{f}(0, x) = {^*f}(x), {^*\bar{g}}(0, x) = {^*g}(x).
$$

The conclusion is immediate.

§4 Examples

It is clear that (0–4) and (0–5) are true for locally infinite poset in the sense of Corollary (3.2). As other interesting application of Theorem 3.3, we give anew another combinatorial proof of the fundamental theorem of the infinitesimal calculus.

Definition 4.1 *By the $p-$cardinality of a partition $\Delta \in \prod(S)$, the lattice of partitions of S, denoted by $\| \Delta \|$, we mean the number of blocks of the partition Δ. Each of blocks of the partition Δ is said to have $p-$cardinality 1. The union U of some blocks of partititons Δ is said to have $p-$counting measure $\bar{\mu}_p(U)$ defined by $\bar{\mu}_p(U) = \| U \| / \| \Delta \|$, where $\| U \|$ means the number of blocks contained in U. $\bar{\mu}_p(U)$ in general is a hyperreal number in R. We define a real-valued $p-$copunting measure by $\mu_p(U) = st(\bar{\mu}_p(U)) = st(\| U \| / \| \Delta \|)$.*

Let K be real field R, and let f be continuous on $[0,a] \subseteq R$. Take $x \in [0,a]$. $[0,x] = S$ is a linearly ordered set which is ordered by usual ordering $\leq$. Make a fine partition $\Delta \in \prod([0,x])$, which is a set of subintervals (x_{i-1}, x_i) $(i = 1, 2, \cdots, v, v \in {}'N \setminus N, x_0 = 0, x_v = x)$ of $[0,x]$ such that the endpoints contain all standard points of $[0,x]$, and

$$x_{i-1} \simeq x_i, \frac{x_i - x_{i-1}}{1/v} \simeq 1 \ (i = 1, 2, \cdots, v).$$

It is a fact that $(x_0, x_1, x_2, \cdots, x_v)$ is just C in Theorem 3.3 and we have

$$\mu_p([x_{i-1}, x_i)) = \mathrm{st}(\bar{\mu}_p([x_{i-1}, x_i))) = \mathrm{st}(\| x_{i-1}, x_i) \| / \| \Delta \|) = \mathrm{st}(1/2) = 0.$$

Suppose ${}^*\mu_1 = \bar{\mu}_p([x_i - 1, x_i))^*\varsigma, {}^*\mu_2 = [1/\bar{\mu}_p(x_{i-1}, x_i)]^{-1*}\varsigma^{-1}$. Then we obatin by Theorem 3.3 that for $x \in [0,a]$,

$$\begin{aligned} g(x) &= {}^*g(x) = \mathrm{st}({}^*g(x)) = \mathrm{st}(\frac{1}{v} \sum_{y \in C \cap {}^*[0,x]} {}^*f(y)^*\varsigma(y,x)) \\ &= \mathrm{st}(\frac{1}{v} \sum_{y \in C \cap {}^*[0,x]} {}^*f(y)). \end{aligned}$$

Since $(x_{i-1}, x_i)/(1/v) \simeq 1, i = 1, \vdots, v$ and f is uniformly continuous on $[0,x]$, it follows from the quasi-Duhamel Principle [11] that

$$g(x) = {}^* g(x) = \mathrm{st}(\frac{1}{v} \sum_{y \in C \cap {}^*[0,x]} {}^*f(y)) = \mathrm{st}(\sum_{i=1}^{v} {}^*f(x_i)(x_i - x_{i-1})) = \int_0^x f(t)\,dt$$

and $g(x)$ is continuous.

It is known that for all $a, b \in C$,

$$\begin{aligned} {}^*\mu(a,a) &= 1/{}^*\varsigma(a,a) = 1, \\ {}^*\mu(a,b) &= -\sum_{a \leq z < b} {}^*\mu(a,z) = -\sum_{a < z \leq b} {}^*\mu(z,b). \end{aligned}$$

By Theorem 3.3, we have

$$\begin{aligned} f(x) &= {}^*f(x) = \mathrm{st}({}^*f(x)) = \mathrm{st}[(\frac{1}{v})^{-1} \sum_{y \in C \cap {}^*[0,x]} {}^*g(y)^*\mu(y,x)] \\ &= \mathrm{st}[(\frac{1}{v})^{-1} \sum_{i=1}^{v} {}^*g(x_i)^*\mu(x_v, x_v)] = \mathrm{st}[(\frac{1}{v})^{-1}({}^*g(x_v) - {}^*g(x_{v-1})] \\ &= \mathrm{st}[(\frac{1}{v})^{-1}({}^*g(x_v) - {}^*g(x_v - \frac{1}{v}))] = \frac{d}{dx} g(x). \end{aligned}$$

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星型关联代数与星式Möbius反演

徐利治　　　孙广润

（大连理工大学，116024）　（曲阜师范大学，273165）

摘　要

本文借助于非标准组合论中的星型有限结构，定义了星型关联代数，从而建立了局部星型有限集上的 Möbius 反演. 由此在一个超结构扩大中，在非标准意义下，将 Möbius 反演推广到局部标准无限半序集上. 文中几例显示，在非标准领域里，本文结果为探索离散数学与连续数学的某些反演的统一性提供了一种可能途径.