Journal of Mathematical Research & Exposition
Vol.12, No.4, Nov. 1992

Isomorphic Decomposition of Complete Graphs into Graceful Forests*

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Abstract. In 1985, S.Ruiz [3] generalized Kirkman's theorem to linear forests (graph each component of which is a path). It is the object of the paper to generalize Kirkman's Theorem to a class of graceful forests.

Throughout the present paper we shall only be concerned with simple graph, that is, graphs in which a pair of G vertices is incident with at most one edge and no loops are admitted. The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively.

If G is a graph and $H_1, H_2, \cdots, H_n (n \geq 2)$ are nonempty, pairwise edge-desjoint subgraph of G having the property that

$$E(G) = \bigcup_{i=1}^{n} E(H_i),$$

then we say that G is the edge sum of $H_1, H_2, \cdots, H_n$ and wirte

$$G = H_1 \oplus H_2 \oplus \cdots \oplus H_n \tag{1}$$

In such a case, we also say that G can be decomposed into the subgraphs $H_1, H_2, \cdots, H_n$. If there is a graph H that is isomorphic to each of the subgraphs $H_1, H_2, \cdots, H_n$, then expression (1) is called a H- decomposition of G, the graph G is said to be H-decomposable, and H is an isopart of. (For the meanings of terms not defined herein, see [1])

In 1947, Kirkman[2] discovered a combinatorial theorem which, when expressed in graph theory terminology, stipulates that the complete gtaph K_{2n} is 1-factorable, or equivalently, K_{2n} is nK_2-decomposable for every $n \geq 1$. In 1985, S.Ruiz[3] generalized Kirkman's theorem to linear forests (graph each component of which is a path).

It is the object of the paper to generalize Kirkman's theorem to caterpillar forests.

A tree T is gracefully numbered when each vertex v is assigned a value $f(v)$, each edge uv is assigned the value $f(uv) = |f(u) - f(v)|$, and the set equations

$$\begin{aligned} \{f(v) \mid v \in V(G)\} &= \{1, 2, \ldots, p\}, \\ \{f(uv) \mid uv \in E(G)\} &= \{1, 2, \ldots, p-1\} \end{aligned}$$

*Received Nov.23 1989.

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hold, where p is the order of T. A tree T is graceful if it admits a graceful numbering. A forest F is graceful if a new graceful tree was obtained by stringing all components of F. A tree T is caterpillar, if T be a tree with all vertices either on a single certral path, or distance 1 away from it. If each component of forest F is a caterpillar tree, then F is said a caterpillar forest. Obviously, caterpillar forest is graceful.

Example The forest F is graceful, it's all components can be stringed into a graceful tree (Fig.1)

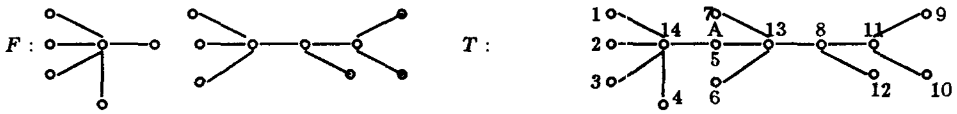

*Fig.*1

The vertex A is called repeated vertex.

Evidently, the linear forests is a special case of the caterpillar forests.

Theorem 1 *If F is a caterpillar forest of size n without isolated vertices, then K_{2n+1} is F-decomposable.*

Proof Assume that the caterpillar forest is given by

$$F = T_{n_1+1} \cup T_{n_2+1} \cup \cdots \cup T_{n_l+1},$$

where T_{n_i+1} is a caterpillar, and $|E(T_{n_i+1})| = n_i, n_1 + n_2 + \cdots + n_l = n$.

Let $T(V, E)$ be the new caterpillar tree we obtain by stringing $T_{n_1+1}, T_{n_2+1}, \cdots, T_{n_l+1}$, then $|V| = n+1, |E| = n$. Since T is a graceful tree, without loss of generality, suppose a graceful numbering of T is the following:

$$v_1 = 1, v_2 = 2, \cdots, v_k = k, \cdots, v_{n+1} = n+1.$$

Now we construct a $(2n+1) \times (2n+1)$ circulant matrix A. The first of A is

$v_1 v_2 \cdots v_n v_{n+1} \underbrace{00 \cdots 0}_{n \text{ terms}}$.

$$A = \begin{pmatrix} v_1 & v_2 & \cdots & v_n & v_{n+1} & 0 & 0 & \cdots & 0 \\ 0 & v_1 & v_2 & \cdots & v_n & v_{n+1} & 0 & \cdots & 0 \\ & & \cdots & & & & \cdots & & \\ 0 & 0 & \cdots & 0 & v_1 & v_2 & \cdots & v_n & v_{n+1} \\ v_{n+1} & 0 & \cdots & 0 & 0 & v_1 & v_2 & \cdots & v_n \\ & & \cdots & & & & \cdots & & \\ v_2 & v_3 & \cdots & v_{n+1} & 0 & \cdots & 0 & 0 & v_1 \end{pmatrix}.$$

In other words, ith row of A is obtained from the first row of A by a cyclic shift of $i-1$ steps, and so any circulant matrix is determined by the first row.

Since $\{v_1, v_2, \cdots, v_{n+1}\}$ is a graceful numbering of T, and by the construction of A, it is easy to see that in each pair of distinct columns of A there is exactly a pair of vertices is joined by an edge. Furthermore, we observe that

(i) There are exactly $n+1$ vertices of T in each row of A.

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(ii) There are exactly $n+1$ vertices $v_1, v_2, \cdots, v_{n+1}$ of T in each column of A, and $d(v_1)+d(v_2)+\cdots+d(v_{n+1})=2n$.

Now, if all vertices in each column of A are identified, we shall obtain a simple graph G in which each pair distinct vertices is joined by an edge, and $d(v)=2n$, for each vertex $v \in G$. Therefore $G=K_{2n+1}$,

$$K_{2n+1}=G=T \oplus T \oplus \cdots \oplus T \quad (2n+1 \text{ terms}).$$

Since T is a caterpillar tree, we can give a graceful numbering of T such that $d(v_1)=d(1)=1$, and the vertex v_1 is not a repeated vertex.(Fig.1)

Without loss of generality, we may assume that the vertex v_1 belongs the component T_{n_1+1}.

Let l be an even integer. By the matrix A, we construct the forest $F_{(1,n+1)}$ as follows:

$$F_{(1,n+1)}=T_{n_1+1} \cup T'_{n_2+1} \cup \cdots \cup T_{n_{l-1}+1} \cup T'_{n_l+1}$$

where $T_{n_1+1}, T_{n_3+1}, \cdots, T_{n_{l-1}+1}$ belong to the first row of A, components $T'_{n_2+1}, T'_{n_4+1}, \cdots, T'_{n_l+1}$, belong to $(n+1)$th row of A. Similarly we obtained forests $F_{(2,n+2)}, \cdots, F_{(n+1,2n+1)}$, $F_{(n+2,1)}, \cdots, F_{(n+2,1)}, \cdots, F_{(2n+1,n)}$.

By the construction of the matrix A and F is a caterpillar forest without isolated vertices, we have that

$$F_{(i,j)} \cong F \quad (i,j=1,2,\cdots,2n+1).$$

Therefor,

$$\begin{aligned} K_{2n+1} &= T \oplus T \oplus \cdots \oplus T \\ &= F_{(1,n+1)} \oplus F_{(2,n+2)} \oplus \cdots \oplus F_{(2n+1,n)} \\ &= F \oplus F \oplus \cdots \oplus F \quad (2n+1) \text{ terms}. \end{aligned}$$

The results in the case of odd l can be proved similarly.

Consequently theorem 1 is proved.

Example A forest with 6 vertices and 4 edges $(n=4)$. The new caterpillar tree T we obtain by stringing all components of F, and the tree T is gracefully numbered:(Fig.2)

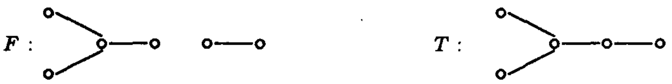

Fig.2

$$A=\begin{array}{c} \begin{array}{ccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \end{array} \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 4 & 5 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

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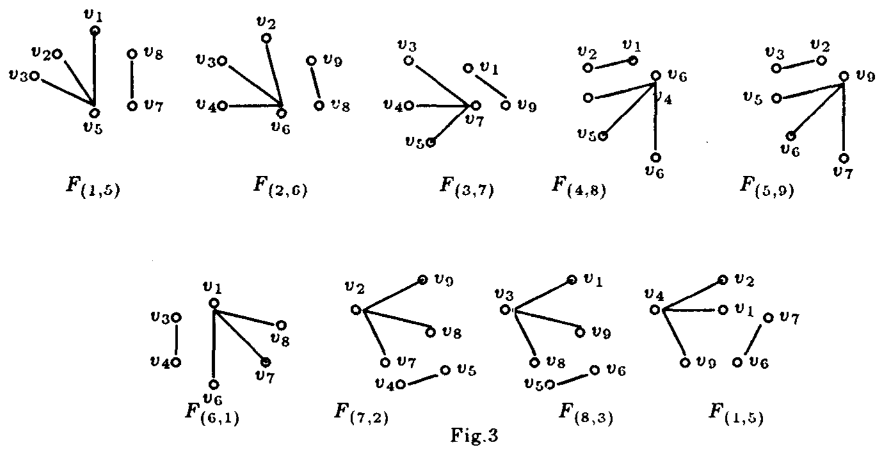

Fig.3

Hence

$$
\begin{aligned}
K_9 &= F_{(1,5)} \oplus F_{(2,6)} \oplus F_{(3,7)} \oplus F_{(4,8)} \oplus F_{(5,9)} \oplus F_{(6,1)} \oplus F_{(7,2)} \oplus F_{(8,3)} \oplus F_{(9,4)} \\
&= F \oplus F \oplus \cdots \oplus F \quad \text{(9 terms.)}
\end{aligned}
$$

Theorem 2 *If F is a caterpillar forest of size n without isolated vertices, then K_{2n} is F-decomposable.*

Proof Assume that the caterpillar forest is given by

$$F = T_1 \cup T_2 \cup \cdots \cup T_s \quad \sum_{i=1}^{s} |E(T_i)| = n.$$

Let $T(V, E)$ be the new caterpillar tree we obtain by stringing $T_1, T_2, \cdots, T_s$, then $|V| = n+1, |E| = n$. Since T is a caterpillar tree, without loss of generality, suppose a graceful numbering of T is the following

$$v_1 = 1, v_2 = 2, \cdots, v_{n+1} = n+1.$$

Since T is a caterpillar tree, we can give a graceful numbering of T such that $d(v_1) = d(1) = 1$.

Now we construct a $(2n-1) \times 2n$ mateix A having the property that the entries in the first column are v_1, and the entries in ith column $(i = 2, 3, \ldots, 2n)$ form a $(2n-1) \times (2n-1)$ circulant matrix A_1. The first row of A_1 is

$$v_2, v_3, \cdots, v_n v_{n+1} \quad \underbrace{00\cdots0}_{(n-1)\text{ terms}}$$

$$
A = \begin{pmatrix}
v_1 & v_2 & v_3 & \cdots & v_{n+1} & 0 & \cdots & & & 0 \\
v_1 & 0 & v_2 & v_3 & \cdots & v_{n+1} & 0 & \cdots & & 0 \\
 & & \cdots & & & & \cdots & & & \\
v_1 & 0 & \cdots & 0 & v_2 & v_3 & \cdots & & & v_{n+1} \\
v_1 & v_{n+1} & 0 & \cdots & 0 & v_2 & \cdots & & & v_n \\
 & & \cdots & & & & \cdots & & & \\
v_1 & v_4 & v_5 & \cdots & v_{n+1} & 0 & 0 & \cdots & v_2 & v_3 \\
v_1 & v_3 & v_4 & \cdots & v_n & v_{n+1} & 0 & \cdots & 0 & v_2
\end{pmatrix}
= \begin{pmatrix}
v_1 & \\
v_1 & \\
\vdots & A_1 \\
v_1 &
\end{pmatrix}
$$

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By virtue of the matrix A, we can prove that K_{2n} is F-decomposable by the same argument in the proof of theorem 1.

References

[1] J.A.Bondy and U.S.R.Murty, *Graph Theory with Application,* Macmillan Press LTD, 1976.

[2] T.P.Kirkman, *On a problem in combinations,* Cambridge and Dublin Math. J., 2(1847)191-204.

[3] S.Ruiz, *Isomorphic decomposition of complete graphs into linear forests,* J.Graph Theory, 9(1985)189-191.

[4] F.Harary and W.Robison, *Isomorphic factorizations X: Unsolved problems,* J.Graph Theory 9(1985)68-69.

完全图对于优美林的同构分解

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摘 要

组合论中著名的 Kirkman 定理用图论的语言可叙述为:完全图 K_{2n} 是可 nK_2 分解的[2]. 1985 年 S. Ruiz 把 Kirkman 定理推广到线性林[3]. 我们进一步把 Kirkman 定理推广到一类优美林.

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