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A Generalization of Algebraic Lattices*

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In this paper, we prove two theorems on lattices and operators.

Definition 1 Let L be a complete lattice. An element a in L is $(*)-$compact if and only if whenever $a \leq VA$ for $A \subseteq L$ then $a \leq VB$ for some $B \subseteq A$ with B is at most countable. L is called $(*)-$algebraic lattice if every element in L is a supremum of $(*)-$compact elements.

$(*)-$algebraic lattice is a generalization of algebraic lattice.

Definition 2 If we are given a set A, a mapping $\phi : P(A) \to P(A)$, where $P(A) = \{S : S \subseteq A\}$ = power set of A, is called a closure operator on A if, for $X, Y \subseteq A$, it satisfies: (1). $X \subseteq \phi(X)$, (2). $\phi(\phi(X)) = \phi(X)$, (3) $X \subseteq Y$ implies $\phi(X) \subseteq \phi(Y)$. A subset X of A is called a closed subset if $\phi(X) = X$. The poset of closed subsets of A with set inclusion as the partial ordering is denoted by $L(\phi)$. A closure operator ϕ on the set A is a $(*)-$operator if for every $X \subseteq A$

$$\phi(X) = \cup\{\phi(Y) : Y \subseteq X \text{ and } Y \text{ is at most countable}\}. \tag{1}$$

Theorem 1 *If ϕ is a $(*)-$operator on a set A, then $L(\phi)$ is a $(*)-$algebraic lattice.*

Proof First we will show that $\phi(X)$ is $(*)-$ compact element if X is at most countable. So suppose $X = \{a_1, a_2, \cdots, a_j, \cdots,\}$ and $\phi(X) \subseteq V_{i\in I}\phi(A_i) = \phi(\cup_{i\in I}A_i) = \cup\{\phi(Y) : Y \subseteq \cup_{i\in I}A_i$ and Y is at most countable$\}$. For each $a_j \in X, X \subseteq \phi(X)$, we have $Y_j \subseteq \cup_{i\in I}A_i$ with $a_j \in \phi(Y_j), Y_j$ is at most countable, since Y_j is at most countable, then there are at most countable many $A_i's$, say $A_{j1}, A_{j2}, \cdots, A_{jn}, \cdots, (A_{jn} \in \{A_i : i \in I\}$ such that $Y_j \subseteq \cup_{n=1}^{\infty}A_{jn}, a_j \in \phi(\cup_{n=1}^{\infty}A_{jn}), X \subseteq \cup_{j=1}^{\infty}\phi(\cup_{n=1}^{\infty}A_{jn}) \subseteq \phi(\cup_{j=1}^{\infty}\cup_{n=1}^{\infty}A_{jn}), \phi(X) \subseteq \phi(\cup_{j,n=1}^{\infty}A_{jn}) = V_{j,n=1}^{\infty}\phi(A_{jn}), A_{jn} \in \{A_i, i \in I\}$. So $\phi(X)$ is $(*)-$compact. Now suppose $\phi(X)$ is not equal to $\phi(Y)$ for any at most countable Y. From $\phi(X) = \cup\{\phi(Y) : Y \subseteq X$ and Y is at most countable$\}$,it is easy to see that $\phi(X)$ cannot be contained in any at most countable union of the $\phi(Y)'s$ (by reduction to absurdity), hence $\phi(X)$ is not $(*)-$compact.For each $\phi(X) \in L(\phi), \phi(X) = \cup\{\phi(Y) : Y \subseteq X \text{and} Y \text{ is at most countable}\}$,

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$\phi(X) = \phi(\cup\{Y : Y \subseteq X \text{and} |Y| \leq \aleph_0\}) = \vee\{\phi(Y) : Y \subseteq X \text{and} |Y| \leq \aleph_0\}, \phi(Y)'s$ are $(*)-$compact, $\phi(Y)'s$ are $(*)-$compact, so $L(\phi)$ is $(*)-$ algebraic lattice. □

Theorem 1' *Every $(*)-$algebraic lattice is isomorphic to the lattice of closed subsets of some set A with a $(*)-$operator.*

proof Let L be a $(*)-$algebraic lattice, and A be the subset of $(*)-$ compact elements. For $X \subseteq A$ define $\phi(X) = \{a \in A : a \leq \vee X\}$.$\phi$ is a $(*)-$ operator. The map $a \longmapsto \{b \in A : b \leq a\}$ gives the isomorphism. □

Theorem 2 *Let $(L, \leq)$ be a complete lattice, $S \subseteq L, S$ is a complete lattice under $\leq$if and only if there is an idempotent isotone self-mapping ψ of L(i.e.$\psi : L \longrightarrow L$, for $x, y \in L, x \leq y$ implies $\psi(x) \leq \psi(y); \psi(\psi(x)) = \psi(x))$ such that $L(\psi) = range(\psi) = S$(i.e., $Fix(\psi) = \{x \in L : \psi(x) = x\} = Image(\psi) = \{\psi(x) : x \in L\} = S$).*

The proof of theorem 2 is very difficult.

Theorem 2' *Let $(L, \leq)$be a complete lattice, $S \subseteq L, S$ is a $(*)-$algebraic lattice under $\leq$ if and only if there are: (1) $\psi : L \longrightarrow S, \psi$ is an idempotent isotone mappmg such that $L(\psi) = range(\psi) = S$; and (2) a $(*)-$ operator ϕ for some $A \subseteq S$ such that $(L(\phi), \subseteq)$ is isomorphic $(S, \leq)$.*

Proof By Theorem 2, Theorem 1 and Theorem 1'.

Note (a). In condition (1), "at mast countable" change over to arbitrary Aleph, then theorem 1 and 1' hold also.

(b). Condition (1) is different from that in Grätger's book.

(c). Theorem 2 is an interesting new theorem on complete lattice, but its proof is very difficult.

References

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