Journal of Mathematical Research & Exposition
Vol.13, No.1, Feb. 1993

On Problem 24 of P. Turán, II*

Shi Yingguang
(Computing Center, Chinese Academy of Sciences, Beijing)

Abstract In this paper we obtain a beter answer to Problem 24 of P. Turán [1]: If the Hermite-Fejer interpolation process converges for any $f \in C[-1,1]$ then the Lagrange interpolation process defined on the same nodes converges for each f with $E_n(f) = o(n^{-\frac{23}{18}})$, where $E_n(f)$ is the deviation of best uniform approximation of $f \in C[-1,1]$ on $[-1,1]$ by polynomials of degree $\le n$.

1. Introduction and Results

Consider a triangular matrix X of nodes

$$-1 \le x_{n1} < x_{n2} < \cdots < x_{nn} \le 1, \quad n = 1, 2, \ldots \tag{1}$$

and sometimes for simplicity omit the superfluous notations. Write for $f \in C[-1,1]$ and for each fixed n($\|\cdot\|$ stands for the Chebyshev norm)

$$\omega_n(x) = (x - x_1)(x - x_2)\cdots(x - x_n);$$

$$l_k(x) := l_{nk}(x) = \frac{\omega_n(x)}{(x - x_k)\omega_n'(x_k)}, \ k = 1, 2, \ldots, n;$$

$$L_n(f, x) = \sum_{k=1}^{n} f(x_k) l_k(x);$$

$$A_k(x) = [1 - \frac{\omega_n''(x_k)}{\omega_n'(x_k)}(x - x_k)] l_k^2(x) := v_k(x) l_k^2(x), \ k = 1, 2, \ldots, n;$$

$$B_k(x) = (x - x_k) l_k^2(x), \ k = 1, 2, \ldots, n;$$

$$H_n(f, x) = \sum_{k=1}^{n} f(x_k) A_k(x);$$

$$\mu_n = \left\| \sum_{k=1}^{n} |A_k(x)| \right\|; \quad \nu_n = \left\| \sum_{k=1}^{n} |B_k(x)| \right\|; \quad \Lambda_n = \left\| \sum_{k=1}^{n} l_n^2(x) \right\|; \quad \lambda_n = \left\| \sum_{k=1}^{n} |l_k(x)| \right\|.$$

To draw a general conclusion from the behavior of the polynomials $H_n(f, x)$ on those of $L_n(f, x)$, P. Turán proposed his Problem 24 [1]:

*Received Apr. 21, 1990

Is it true that, for any matrix X satisfying

$$\lim_{n\to\infty} \|H_n(f,x) - f(x)\| = 0, \ \forall f \in C[-1,1], \tag{2}$$

we have

$$\lim_{n\to\infty} \|L_n(f,x) - f(x)\| = 0 \tag{3}$$

for all functions f which are continuously differentiable in $[-1,1]$?

At first Vértesi [2] deduces the following estimates (Here and later the signs"$\bigcirc$" and "$\circ$" depend only on X).

Theorem A *For any matrix X*

(a) $\nu_n = \bigcirc(n^3\mu_n)$;

(b) $\lambda_n = \bigcirc(n^{\frac{5}{2}}(\ln^{\frac{1}{2}} n)\mu_n)$;

(c) If $\mu_n = \bigcirc(1)$,

$$\lambda_n = \bigcirc(n^{\frac{5}{2}}).$$

Based on this theorem he gets the first answer to Problem 24 of P. Turán without further assumption (e.g., ρ-normality, etc.), since (2) implies that $\mu_n = \bigcirc(1)$.

Theorem B *If $\mu_n = \bigcirc(1)$, then (3) holds for all $f \in \{f : E_n(f) = \circ(n^{-\frac{5}{2}})\}$, where $E_n(f)$ is the deviation of best uniform approximation of $f \in C[-1,1]$ on $[-1,1]$ by polynomials of degree $\leq n$.*

Later, the author in [3] deduces the estimate

$$\left\| \sum_{k=1}^{n} |(x - x_k) l_k^2(x)| \right\| = \bigcirc(n^{\frac{1}{2}}\mu_n^{\frac{1}{2}})$$

and hence finds a better answer to the problem.

Theorem C *If $\mu_n = \bigcirc(1)$, then (3) holds for all $f \in \{f : E_n(f') = \circ(n^{-\frac{1}{2}})\}$,*

Now in this paper we first intend to give the following two lemmas which improve the estimates in Theorem A.

Lemma 1 *For any matrix X: (a)* $\max_{1\leq k\leq n} \|l_{nk}\| = \bigcirc(n^{\frac{2}{3}}\mu_n^{\frac{1}{2}})$; *(b)* $\Lambda_n = \bigcirc(n^{\frac{7}{3}}\mu_n)$; *(c)* $\nu_n = \bigcirc(n^{\frac{7}{6}}\mu_n)$; *(d)* $\lambda_n = \bigcirc(n^{\frac{5}{3}}\mu_n^{\frac{1}{2}})$.

Lemma 2 *If $\mu_n = \bigcirc(1)$, then (a)* $\Lambda_n = \bigcirc(n^{\frac{28}{15}})$; *(b)* $\nu_n = \bigcirc(n^{\frac{14}{15}})$; *(c)* $\lambda_n = \bigcirc(n^{\frac{23}{18}})$.

Based on this lemma one can deduce a better answer to the problem than before.

Theorem *If $\mu_n = \bigcirc(1)$, then (3) holds for all $f \in \{f : E_n(f) = \circ(n^{-\frac{23}{18}})\}$.*

2. Proofs

2.1 Proof of Lemma 1

(a) In [3] the author gives an identity

$$\sum_{k=1}^{n}(x-x_k)^2A_k(x)=2\sum_{k=1}^{n}(x-x_k)^2l_k^2(x)$$

from which it follows that

$$\left\|\sum_{k=1}^{n}(x-x_k)^2l_k^2(x)\right\|=\frac{1}{2}\left\|\sum_{k=1}^{n}(x-x_k)^2A_k(x)\right\|\le 2\left\|\sum_{k=1}^{n}|A_k(x)|\right\|=2\mu_n. \tag{4}$$

We intend to show a stronger estimate

$$\max_{1\le k\le n}\|l_{nk}\|\le 8n^{\frac{2}{3}}\mu_n^{\frac{1}{2}},\ n=1,2,\cdots.$$

Suppose on the contrary that there exist numbers n and $k, 1\le k\le n$, such that

$$\|l_{nk}\|>8n^{\frac{2}{3}}\mu_n^{\frac{1}{2}}.$$

Choose $z_n\in[-1,1]$ so that $|l_{nk}(z_n)|=\|l_{nk}\|$. It follows from (4) that

$$|z_n-x_k|<4^{-1}n^{-\frac{2}{3}}. \tag{5}$$

By definition we obtain

$$|v_k(z_n)|<8^{-2}n^{-\frac{4}{3}}. \tag{6}$$

Let $t_n\in[-1,1]$ satisfy $|t_n-z_n|=8^{-1}n^{-2}$.

Noting that $v_k(x)$ is linear and $v_k(x_k)=1$ we have

$$\frac{v_k(t_n)-v_k(z_n)}{t_n-z_n}=\frac{v_k(x_k)-v_k(z_n)}{x_k-z_n}=\frac{1-v_k(z_n)}{x_k-z_n}$$

and hence

$$|v_k(t_n)|=\frac{1}{|x_k-z_n|}|t_n-z_n+(x_k-t_n)v_k(z_n)|. \tag{7}$$

Using the inequality

$$|x_k-t_n|\le|x_k-z_n|+|z_n-t_n|\le 4^{-1}n^{-\frac{2}{3}}+8^{-1}n^{-2}\le 2^{-1}n^{-\frac{2}{3}},$$

one obtains by (5)-(7)

$$|v_k(t_n)|\ge 4n^{\frac{2}{3}}|8^{-1}n^{-2}-(2^{-1}n^{-\frac{2}{3}})(8^{-2}n^{-\frac{4}{3}})|=\frac{15}{32}n^{-\frac{4}{3}}.$$

On the other hand, by the Markov's inequality

$$\begin{aligned}l_k^2(t_n) &= l_k^2(z_n)+\int_{z_n}^{t_n}[l_k^2(t)]'dt\ge\|l_k^2\|-4(n-1)^2\|l_k^2\||t_n-z_n|\\ &\ge \|l_k^2\|\left[1-\frac{4(n-1)^2}{8n^2}\right]\ge\frac{1}{2}\|l_k^2\|>32n^{\frac{4}{3}}\mu_n.\end{aligned}$$

Thus $|v_k(t_n)|l_k^2(t_n) > 15\mu_n$, a contradiction.

(b) The above conclusion implies

$$\Lambda_n = \left\|\sum_{k=1}^{n} l_k^2\right\| = O(n(n^{\frac{2}{3}}\mu_n^{\frac{1}{2}})^2) = O(n^{\frac{7}{3}}\mu_n).$$

(c) By the Cauchy's inequality we have

$$\nu_n = \left\|\sum_{k=1}^{n} |x - x_k| l_k^2(x)\right\| \le \left\|\left\{\sum_{k=1}^{n}(x - x_k)^2 l_k^2(x)\right\}^{\frac{1}{2}} \left\{\sum_{k=1}^{n} l_k^2(x)\right\}^{\frac{1}{2}}\right\| = O(n^{\frac{7}{6}}\mu_n).$$

(d) The conclusion (b) implies $\lambda_n = O(n^{\frac{5}{3}}\mu_n^{\frac{1}{2}})$.

2.2 Proof of Lemma 2

(a) Let n be fixed and let $\sum l_k^2(z_n) = \Lambda_n$ for $z_n \in [-1,1]$. Assume that $\beta, 0 < \beta < 1$, is a fixed number which will be determined later. Since

$$2\mu_n \ge \sum_{|x_k - z_k| \ge n^{-\beta}} (z_n - x_k)^2 l_k^2(z_n) \ge n^{-2\beta} \sum_{|x_k - z_k| \ge n^{-\beta}} l_k^2(z_n),$$

we have

$$\sum_{|x_k - n_k| \ge n^{-\beta}} l_k^2(z_n) = O(n^{2\beta}).$$

Denote by $M_n\{K\}$ the number of elements in the set K.

Here we must apply Theorem 3.4 in [2] which says that denoting by $N_n(\alpha_n, \beta_n)$ the number of $\theta_{nk} = \arccos x_{nk}$ in the interval $[\alpha_n, \beta_n] \subset [0, \pi]$ one has

$$N_n(\alpha_n, \beta_n) = \frac{\beta_n - \alpha_n}{\pi} n + O(\ln n \cdot \ln(n\mu_n)).$$

Put

$$a_n = \min\{1, z_n + n^{-\beta}\}, b_n = \max\{-1, z_n - n^{-\beta}\}, \alpha_n = \arccos a_n, \beta_n = \arccos b_n.$$

Then using the inequality

$$\theta = \pi \cdot \frac{2}{\pi}(\frac{\theta}{2}) \le \pi \sin\frac{\theta}{2} = \pi[\frac{1}{2}(1 - \cos\theta)]^{\frac{1}{2}},\ 0 \le \theta \le \pi$$

we obtain

$$\begin{aligned}
\beta_n - \alpha_n &= \arccos b_n - \arccos a_n = \int_{b_n}^{a_n} (1 - t^2)^{-\frac{1}{2}} dt \\
&\le \int_{1-(a_n - b_n)}^{1} (1 - t^2)^{-\frac{1}{2}} dt = \arccos[1 - (a_n - b_n)] \\
&\le \arccos[1 - 2n^{-\beta}] \le \pi\{\frac{1}{2}[1 - (1 - 2n^{-\beta})]\}^{\frac{1}{2}} \\
&= \pi n^{-\frac{\beta}{2}}.
\end{aligned}$$

Thus

$$\begin{aligned} M_n: &= M_n\{k : |x_{kn} - z_n| \le n^{-\beta}\} = N_n(\alpha_n, \beta_n) \\ &\le \pi n^{-\frac{\beta}{2}} \cdot \frac{n}{\pi} + \bigcirc(\ln^2 n) = \bigcirc(n^{1-\frac{\beta}{2}}). \end{aligned}$$

Hence we get an estimate by Lemma 1

$$\sum_{|x_k - z_n| < n^{-\beta}} l_k^2(z_n) = \bigcirc(M_n n^{\frac{4}{3}}) = \bigcirc(n^{\frac{7}{3}-\frac{\beta}{2}}).$$

Thus

$$\sum l_k^2(z_n) = \bigcirc(n^{2\beta}) = \bigcirc(n^{\frac{7}{3}-\frac{\beta}{2}}).$$

Taking $\beta = \frac{14}{15}$, one obtains an estimate $\Lambda_n = \bigcirc(n^{\frac{28}{15}})$.

(b) Using the same argument as the proof of Lemma 1(c) it is easy to get

$$\nu_n = \bigcirc(n^{\frac{14}{15}}).$$

(c) By the Cauchy's inequality we have

$$\left\| \sum_{k=1}^{n} |(x - x_k) l_k(x)| \right\| \le \left\| \left\{ \sum_{k=1}^{n} (x - x_k)^2 l_k^2(x) \right\}^{\frac{1}{2}} \left\{ \sum_{k=1}^{n} 1^2 \right\}^{\frac{1}{2}} \right\| = \bigcirc(n^{\frac{1}{2}})$$

from which it follows by a similar argument as above that

$$\sum_{|x_k - z_n| \ge n^{-\beta}} |l_k(z_n)| = \bigcirc(n^{\beta+\frac{1}{2}}),$$

where $\sum |l_k(z_n)| = \lambda_n$. Similarly we can get another estimate

$$\sum_{|x_k - z_n| < n^{-\beta}} |l_k(z_n)| = \bigcirc(M_n n^{\frac{2}{3}}) = \bigcirc(n^{\frac{5}{3}-\frac{\beta}{2}}).$$

Thus

$$\lambda_n = \sum |l_k(z_n)| = \bigcirc(n^{\beta+\frac{1}{2}}) + \bigcirc(n^{\frac{5}{3}-\frac{\beta}{2}}) = \bigcirc(n^{\frac{23}{18}})$$

if we take that $\beta = \frac{7}{9}$.

2.3 Proof of Theorem

Let P_n be the best approximation of polynomial of degree $\le n-1$ to f. Then

$$\begin{aligned} |L_n(f,x) - f(x)| &\le |L_n(f,x) - L_n(P_n,x)| + |L_n(P_n,x) - P_n(x)| + P_n(x) - f(x)| \\ &= |L_n(f - P_n, x)| + |P_n(x) - f(x)| \\ &\le E_{n-1}(f)(1 + \lambda_n) = \circ(1) \end{aligned}$$

if $E_n(f) = \circ(1)$.

References

[1] P. Turán, *On some problems of approximation theory,* J. Approx. Theory,**29**(1980),23-85.

[2] P. Vértesi, *Two problems of P. Turán, in "Studies in pure mathematics",* Birkhauser, Basel-Boston, Mass., 1983,743- 750.

[3] Y.G. Shi, *On Problem 24 of P. Turán, in "Progress in Approximation Theory"* (P. Nevai and A. Pinkus, Eds.),pp. 913-916, Academic Press, New York, 1991.

关 于 P. Turán 问 题 24, Ⅱ

史 应 光

(中国科学院计算中心,北京 100080)

摘 要

在本文中我们得到了一个比[1]中更好的 P. Turán 问题 24 的答案:若 Hermite—Fejér 插值过程对于任何 $f\in C[-1,1]$ 都一致收敛,则定义于同一组节点上的 Lagrange 插值过程对于每个 $f\in\{f:E_n(f)=o(n^{-\frac{23}{18}})\}$ 都一致收敛,这里 $E_n(f)$ 为 $f\in C[-1,1]$ 的用次数 $\leqslant n$ 的代数多项式逼近的偏差.