Journal of Mathematical Research & Exposition
Vol.13, No.1, Feb. 1993

# Characterizations of Completely Distributive Lattice*

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In this note, a completely distributive lattice is characterized by means of Galois connections and generalized order-homomorphisms.

The set of all lower sets and the set of all upper sets in a complete lattice $L$ is respectively written Low $(L)$ and Upp $(L)$. Our main results are:

**Theorem 1** *For a complete lattice $L$, the following conditions are equivalent:*

*(1) $L$ is completely distributive.*

*(2) For each $x \in L$, there is a smallest lower set $A$ with* $\sup A \geq x$.

*(3) The sup map $r = (A \to \sup A) : Low(L) \to L$ has a lower adjoint.*

*(4) The sup map $r : Low(L) \to L$ preserves all sups and infs.*

*(5) The map $f = (x \to \cap r^{-1}(\uparrow x)) : L \to Low(L)$ is a generalized order-homomorphism.*

*(6) There is a generalized order-homomorphism $f : L \to Low(L)$ such that $r$ is the inverse of $f$.*

**Corollary** *Let $L$ be a completely distributive lattice. Then the minimal map $\beta : L \to Low(L)$ is a generalized order- homomorphism and its inverse is the sup map $r : Low(L) \to L$.*

Dually we have:

**Theorem 2** *For a complete lattice $L$, the following conditions are equivalent:*

*(1) $L$ is completely distributive.*

*(2) For each $x \in L$, there is a smallest upper set $A$ with $inf A \geq x$.*

*(3) The inf map $s = (A \to inf A) : Upp(L) \to L$ has a upper adjoint.*

*(4) The inf map $s : Upp(L) \to (L)$ preserves all sups add infs.*

*Received Feb. 11, 1991.