Journal of Mathematical Research & Exposition
Vol.13, No.2, May 1993

Finding Some Strange Identities via Faa di Bruno's Formula *

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Abstract Here proposed is a kind of constructive procedure that can be used to find various strange identities involving summations over partitions.

1. Introduction

Throughtout this note the following notations will be used.
$\mathbf{Z}_+$: set of positive integers, and $\mathbf{Z}_0 \equiv \mathbf{Z}_+ \cup \{0\}$.

$\sigma(n)$: set of partitions of n $(n \in \mathbf{Z}_+)$, represented by $1^{k_1}2^{k_2}\cdots n^{k_n}$ with $k_1 + 2k_2 + \cdots + nk_n = n, k_i \in \mathbf{Z}_0$, $(i = 1, 2, \cdots, n)$.

$\bar{k} = (k_1, k_2, \cdots, k_n)$: vector representing the partition $1^{k_1}2^{k_2}\cdots n^{k_n}$.
$k = k_1 + k_2 + \cdots + k_n$: number of parts of the partition $1^{k_1}2^{k_2}\cdots n^{k_n}$.
$(x)_{(k)} = x(x-1)\cdots(x-k+1)$: the k-th falling factorial of x, $(x)_{(0)} = 1$.

$\binom{x}{\bar{k}} = (x)_{(k)}/\Pi_{i=1}^n k_i!$: the multinomial coefficient.

We will investigate various special identities involving summations of the type

$$\sum_{\sigma(n)} F(\bar{k}) \equiv \sum_{\sigma(n)} F(k_1, k_2, \cdots, k_n), \tag{0}$$

where for each specially commound function $F(\bar{k})$ defined on $\mathbf{Z}_0^n$ the summation is taken over all the partitions of n. As examples two well-known particular identities may be mentioned as follows

$$\sum_{\sigma(n)} \frac{1}{k_1!k_2!\cdots k_n!1^{k_1}2^{k_2}\cdots n^{k_n}} = 1, \tag{1}$$

*Received Nov. 26, 1992. Supported by the National Natural Science Foundation of China. The material of this paper was a part of the talk presented at the Dept. of Math. & Statistics, Univ. of Pittsburgh, U.S.A..

$$\sum_{\sigma(n)} \frac{n!}{k_1!k_2!\cdots k_n!(1!)^{k_1}(2!)^{k_2}\cdots(n!)^{k_n}} = B_n, \tag{2}$$

where B_n ($n \in \mathbf{Z}_0$) are Bell numbers that may be generated by

$$\exp(e^t - 1) = \sum_{n\geq 0} \frac{t^n}{n!} \tag{3}$$

and (1) is the very old identity known to Cauchy (cf. [1], [2], [5]).

Let $B_n^{(m)}$, $E_n^{(m)}$ and F_n ($n \in \mathbf{Z}_0$) denote the generalized Bernoulli numbers, generalized Euler numbers and Fibonacci numbers, respectively, which may also be defined by the following gnerating functions (cf. [4])

$$\left(\frac{t}{e^t - 1}\right)^m = \sum_{n\geq 0} \frac{B_n^{(m)}}{n!} t^n, \quad \left(\frac{2e^t}{e^{2t}+1}\right)^m = \sum_{n\geq 0} \frac{E_n^{(m)}}{n!} t^n, \tag{4}$$

$$\frac{1}{1-t-t^2} = \sum_{n\geq 0} F_n t^n. \tag{5}$$

Also, let us recall that the Gegenbauer polynomials $C_n^{(\lambda)}(z)$ and Stirling polynomials $\sigma_n(z)$ may be defined by the following gnerating functions respectively (cf. [3], [4])

$$(1 - 2zt + t^2)^{-\lambda} = \sum_{n\geq 0} C_n^{(\lambda)}(z) t^n, \tag{6}$$

$$\left(\frac{te^t}{e^t - 1}\right)^z = z \sum_{n\geq 0} \sigma_n(z) t^n, \quad (z \neq 0). \tag{7}$$

What we are going to demonstrate are several peculiar identities, most of which involve some complicated summations but give very simple sums, namely

$$\sum_{\sigma(n)} \frac{(-1)^{k-1}}{k} \binom{k}{\bar{k}} \left(\frac{B_1}{1!}\right)^{k_1} \left(\frac{B_2}{2!}\right)^{k_2} \cdots \left(\frac{B_n}{n!}\right)^{k_n} = \frac{1}{n!}, \tag{8}$$

$$\sum_{\sigma(n)} \binom{1/\alpha}{\bar{k}} \binom{\alpha}{1}^{k_1} \binom{\alpha+1}{2}^{k_2} \cdots \binom{\alpha+n-1}{n}^{k_n} = 1, \quad (\alpha \neq 0), \tag{9}$$

$$\sum_{\sigma(n)} \binom{-1/m}{\bar{k}} \left(\frac{B_1^{(m)}}{1!}\right)^{k_1} \left(\frac{B_2^{(m)}}{2!}\right)^{k_2} \cdots \left(\frac{B_n^{(m)}}{n!}\right)^{k_n} = \frac{1}{(n+1)!}, \tag{10}$$

$$\sum_{\sigma(n)} \binom{-1/m}{\bar{k}} \left(\frac{E_1^{(m)}}{1!}\right)^{k_1} \left(\frac{E_2^{(m)}}{2!}\right)^{k_2} \cdots \left(\frac{E_n^{(m)}}{n!}\right)^{k_n} = \begin{cases} 1/n!, & (n: \text{ even}) \\ 0, & (n: \text{ odd}), \end{cases} \tag{11}$$

$$\sum_{\sigma(n)} (-1)^{k-1}(k-1)! \frac{F_1^{k_1} F_2^{k_2} \cdots F_n^{k_n}}{k_1!k_2!\cdots k_n!} = \frac{1}{n} \left\{ \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \right\}, \tag{12}$$

$$\sum_{\sigma(n)}\binom{-1/\lambda}{k}\left(C_1^{(\lambda)}(z)\right)^{k_1}\left(C_2^{(\lambda)}(z)\right)^{k_2}\cdots\left(C_n^{(\lambda)}(z)\right)^{k_n}=\begin{cases}-2z, & \text{for } n=1,\\ 1, & \text{for } n=2,\\ 0, & \text{for } n\geq 3,\end{cases} \tag{13}$$

$$\sum_{\sigma(n)}\binom{-1/z}{k}(z\sigma_1(z))^{k_1}(z\sigma_2(z))^{k_2}\cdots(z\sigma_n(z))^{k_n}=\frac{(-1)^n}{(n+1)!}. \tag{14}$$

Worth noticing is that the sums given by (9), (10), (11), (13) and (14) are entirely independent of the parameters α, m, λ and z contined in their summands respectively.

2. A general procedure

We will formulate a constructive procedure that can be used to obtain all the identities mentioned in §1. First let us state a particularly devised proposition which is implied by Faa di Bruno's formula (cf. [1]).

Proposition *Let $\phi(t)$ be a formal power series written in the form*

$$\phi(t)=\sum_{n\geq 0}\left[\begin{matrix}\phi\\ n\end{matrix}\right]t^n \tag{15}$$

and let $f(x)$ be an infinitely differentiable function with $f^{-1}(x)$ as its compositional inverse so that $f^{-1}\circ f(x)=x$. Suppose that we have a series expansion of the form

$$f\circ\phi(t)=\sum_{n\geq 0}\left[\begin{matrix}f\circ\phi\\ n\end{matrix}\right]t^n. \tag{16}$$

Also, we denote (with the differential operator $D_x=d/dx$)

$$(f^{-1})_0^{(k)}=\left[D_x^k f^{-1}(x)\right]_{x=f\circ\phi(0)}. \tag{17}$$

Then we have the formula

$$\left[\begin{matrix}\phi\\ n\end{matrix}\right]=\sum_{\sigma(n)}(f^{-1})_0^{(k)}\prod_{i=1}^{n}\frac{1}{k_i!}\left[\begin{matrix}f\circ\phi\\ i\end{matrix}\right]^{k_i}. \tag{18}$$

Proof Notice that $\phi(t)=(f^{-1}\circ f)\circ\phi(t)=f^{-1}\circ(f\circ\phi)(t)$ and that the coefficients contained in both (15) and (16) may be expressed in terms of the formal derivatives

$$\left[\begin{matrix}\phi\\ n\end{matrix}\right]=\frac{1}{n!}[D_t^n\phi(t)]_{t=o},\qquad \left[\begin{matrix}f\circ\phi\\ n\end{matrix}\right]=\frac{1}{n!}[D_t^n f\circ\phi(t)]_{t=0}.$$

Hence one can get (18) by just applying Faa di Bruno's formula to the composite function $f^{-1}\circ(f\circ\phi)(t)$.

Procedure of construction Suppose that a certain sequence of special numbers or functions has been defined by a generating function $G(t)$ (a power series in t), and it is required to construct a particular identity of the type (0) in which the summand consists of those given special numbers or functions as factors. Generally one can accomplish this by carrying through the following procedure as suggested by the proposition:

1° Choose suitable $\phi(t)$ and $f(x)$ such that $G(t)$ can be expressed in the form $G(t) = f \circ \phi(t)$, where $\phi(t)$ should be so simple that $[{\phi \atop n}]$ can be explicitly computed, and the compositional inverse $f^{-1}(x)$ should be also easily treated.

2° Compute the numbers $[{\phi \atop n}]$ and $[{f \circ \phi \atop n}]$, $(n \in \mathbf{Z}_+)$.

3° Compute the derivatives $(f^{-1})_0^{(k)} = [D_x^k f^{-1}(x)]_{x=f\circ\phi(0)}, (k \in \mathbf{Z}_+)$.

4° Substitute all the numbers obtained by 2°–3° into (18).

Definition *Once one constructs an identity of the type (0) by means of the above procedure, the set of functions $\{\phi(t), f(x), f^{-1}(x)\}$ will be called the "source" of the identity.*

Thus in order to prove/verify an identity which is actually implied by the proposition, the key step is to discover the source of the identity. In what follows we will mention two typical examples.

Verification of (11) In accordance with the above procedure and in view of the form of the generating function by (4), it should be very natural to choose the following set of functions to be the source of the identity (11):

$$\left\{\phi(t) = (e^{2t}+1)/2e^t, \quad f(x) = x^{-m}, \quad f^{-1}(x) = x^{-1/m}\right\}.$$

Indeed we have

$$\phi(t) = \frac{e^t + e^{-t}}{2} = \sum_{n\geq 0} \frac{t^{2n}}{(2n)!}, \quad [{\phi \atop n}] = \begin{cases} 1/n! & (n: \ \text{even}) \\ 0 & (n: \ \text{odd}) \end{cases}$$

and

$$[{f\circ\phi \atop n}] = \frac{1}{n!} E_n^{(m)}, \quad (n \in \mathbf{Z}_+).$$

Finally we find

$$(f^{-1})_0^{(k)} = \left[D^k(x^{-1/m})\right]_{x=f\circ\phi(0)} = (-1/m)_{(k)}.$$

Hence (11) is obtained by means of the proposition.

Verification of (12) We have to choose the following set of functions

$$\left\{\phi(t) = \log\frac{1}{1-t-t^2}, \ f(x) = e^x, \ f^{-1}(x) = \log x\right\}$$

to be the source of (12). In fact we have then

$$\phi(t) = \log\frac{1}{(1-at)(1-bt)} = \sum_{n\geq 1}\frac{a^n+b^n}{n}t^n,$$

where $a = \frac{1}{2}(1+\sqrt{5}),\ b = \frac{1}{2}(1-\sqrt{5})$. Thus we obtain

$$[{\phi \atop n}] = \frac{a^n + b^n}{n}, \quad [{f\circ\phi \atop n}] = F_n.$$

The final step is to compute

$$(f^{-1})_0^{(k)} = \left[D^k \log x\right]_{x=1} = (-1)^{k-1}(k-1)!.$$

Hence (12) is proved by means of the proposition.

It is not difficult to observe that all the other identities displayed in §1 can be verified by making use of their source shown in the following table

Identity	$\phi(t)$	$f(x)$	$F^{-1}(x)$
(1)	$1/(1-t)$	$\log x$	e^x
(2)	$\exp(e^t-1)$	$\log x$	e^x
(8)	e^t-1	e^x	$\log x$
(9)	$1/(1-t)$	x^α	$x^{1/\alpha}$
(10)	$(e^t-1)/t$	x^{-m}	$x^{-1/m}$
(13)	$1-2zt+t^2$	$x^{-\lambda}$	$x^{-1/\lambda}$
(14)	$\frac{1}{t}(e^t-1)/e^t$	x^{-z}	$x^{-1/z}$

The details may be left to the interested reader.

3. More examples

As is known, the generalized Bernoulli polynomials $B_n^{(m)}(z)$, generalized Euler polynomials $E_n^{(m)}(z)$ and the generalized Laguerre polynomials $L_n^{(\alpha)}(z)$ may be defined by the following generating functions respectively (cf. [4])

$$e^{zt}\left(\frac{t}{e^t-1}\right)^m = \sum_{n\geq 0}\frac{B_n^{(m)}(z)}{n!}t^n, \quad e^{zt}\left(\frac{2}{e^t+1}\right)^m = \sum_{n\geq 0}\frac{E^{(m)}(z)}{n!}t^n,$$

$$(1-t)^{-(\alpha+1)}\exp\left(\frac{zt}{t-1}\right) = \sum_{n\geq 0} L_n^{(\alpha)}(z)t^n, \quad (\alpha > -1).$$

Accordingly, applying the procedure of §2 one may choose the following sources

$$\left\{\phi(t) = e^{-\frac{1}{m}zt}\left(\frac{e^t-1}{t}\right),\ f(x) = x^{-m},\ f^{-1}(x) = x^{-1/m}\right\} \tag{19}$$

$$\left\{\phi(t) = e^{-\frac{1}{m}zt}\left(\frac{e^t+1}{t}\right),\ f(x) = x^{-m},\ f^{-1}(x) = x^{-1/m}\right\} \tag{20}$$

$$\left\{\phi(t) = (\alpha+1)\log\frac{1}{1-t} + \frac{zt}{t-1},\ f(x) = e^x,\ f^{-1}(x) = \log x\right\} \tag{21}$$

to obtain three identities as follows

$$\sum_{\sigma(n)}\binom{-1/m}{k}\left(\frac{B_1^{(m)}(z)}{1!}\right)^{k_1}\cdots\left(\frac{B_n^{(m)}(z)}{n!}\right)^{k_n}=\frac{(-1)^{n+1}}{(n+1)!}\left\{(\frac{z}{m}-1)^{n+1}-(\frac{z}{m})^{n+1}\right\} \tag{22}$$

$$\sum_{\sigma(n)}\binom{-1/m}{k}\left(\frac{E_1^{(m)}(z)}{1!}\right)^{k_1}\cdots\left(\frac{E_n^{(m)}(z)}{n!}\right)^{k_n}=\frac{(-1)^{n}}{2\cdot n!}\left\{(\frac{z}{m}-1)^{n}+(\frac{z}{m})^{n}\right\} \tag{23}$$

$$\sum_{\sigma(n)}\frac{(-1)^{k-1}}{k}\binom{k}{k}\left(L_1^{(\alpha)}(z)\right)^{k_1}\cdots\left(L_n^{(\alpha)}(z)\right)^{k_n}=\frac{\alpha+1}{n}-z. \tag{24}$$

In fact, from the functions given by (19) one easily finds

$$[{\phi \atop n}]=\frac{(-1)^{n+1}}{(n+1)!}\left\{(\frac{z}{m}-1)^{n+1}-(\frac{z}{m})^{n+1}\right\},\quad (f^{-1})_0^{(k)}=(1-/m)_{(k)}.$$

Hence (22) follows from the proposition as a consequence. Certainly, (23) and (24) can be verified in a similar manner.

Finally let us show how to express the polynomial $L_n^{(\alpha)}(z)$ as a finite sum in terms of set partitions. Evidently this can be done using the source

$$\left\{\phi(t)=(\frac{1}{1-t})^{\alpha+1}\exp(\frac{zt}{t-1}),\quad f(x)=\log x,\quad f^{-1}(x)=e^x\right\}.$$

Here the choice of $f(x)$ just leads to an easy computation for $[{f\circ\phi \atop n}]$. Indeed it is easily found that $[{f\circ\phi \atop n}]=\frac{\alpha+1}{n}-z$ and $(f^{-1})_0^{(k)}=[e^x]_{x=0}=1$. Hence we obtain by means of (18)

$$L_n^{(\alpha)}(z)=\sum_{\sigma(n)}\frac{(\frac{\alpha+1}{1}-z)^{k_1}(\frac{\alpha+1}{2}-z)^{k_2}\cdots(\frac{\alpha+1}{n}-z)^{k_n}}{k_1!k_2!\cdots k_n!}. \tag{25}$$

Remark Worth mentioning is that (25) can be used to derive a certain asymptotic expansion of $L_n^{(\lambda\alpha)}(\mu z)$ for $\lambda\uparrow\infty$, $\mu\uparrow\infty$ and $n\uparrow\infty$ with $n=o(|\lambda+\mu|^{1/2})$. In fact, this can be shown by using the same technique as that used in the author's paper (cf. "Power-type generating functions", Colloquia Mathematica Societatis János Boyai, 58. Approximation Theory, Kecskemet (Hungary), p.405–412, 1990) in which the set $\sigma(n)$ is decomposed into subsets $\sigma(n,k)$ (consisting of partitions of n with k parts) and the corresponding summations over $\sigma(n,k)$'s are then treated according to such ordering $k=n,n-1,n-2$, etc. The details will not be discussed here.

Surely, the general procedure of §2 may still be employed to find other special identities.

Acknowledgment *It is a pleasure to thank Prof. Richard Askey, my host at the University of Wissconsin (Madison) during my short visit in August of 1992. The basic idea of this note was actually conceived during this time.*

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经由 Faa di Bruno 公式寻找各种奇异恒等式

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摘要

本文给出寻找一批奇异恒等式的一般方法. 这些在分划集上求和的恒等式包含一些著名的特殊数列及特殊多项式作为被加项因子.