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On Conditional Beleaf Functions *

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Abstract The criteria for establishing conditional beleaf function are suggested, and the advantages and disadvantages of the suggested conditional beleaf functions are discussed. Finally a new formula, which satisfies all the desired conditions, is given for computing conditional beleaf.

§1. Preliminary

Beleaf function is a generalization of probability measure. It was introduced by Dempster [2], and has been developed in great detail by Shfer [5,6,7]. Conditional beleaf functions, just like conditional probability in the theory of probability, is a important part of the theory of beleaf functions. The main goal of this paper is to find out a best formula for computing conditional beleaf.

Let Ω be a nonempty set, J be a *sigma*-algebra of subsets of Ω, and F a real-valued function defined on J. We shall call F a beleaf function on (Ω, J) iff

$(*)$ $F(\emptyset) = 0, F(\Omega) = 1$. $(**)$ If I is a finite subset of the set N of all natural numbers and $\{A\} \cup \{A_i : i \in I\} \subset J$, then

$$\bigcup_{i\in I} A_i \subset A \Longrightarrow F(A) \geq \sum_{\emptyset\neq J\subset I} (-1)^{|J|+1} F(\bigcap_{j\in J} A_j).$$

If this is the case we call (Ω, J, F) a beleaf space.

For a given beleaf space (Ω, J, F), there is a probability space $(X, 2^X, P)$ and a mapping $\Gamma : X \to J$ such that $F(A) = P\{x : \Gamma(x) \subset A\}$. Conversely, if for (Ω, J) there is a probability space $(X, 2^X, P)$ and a set-valued mapping Γ which takes points in X to nonempty subsets of Ω, then the real-valued function F defined above is a beleaf function on (Ω, J) [4]. We call $(X, 2^X, P, \Gamma)$ a source of (Ω, J, F).

Furthermore if Ω is finite, we difine mapping $m : 2^\Omega \to [0,1]$ by $m(A) = P\{x : \Gamma(x) = A\}$. The function m satisfies:

(i) $m(\emptyset) = 0$; (ii) $\sum_{A\subset\Omega} m(A) = 1$; (iii) $F(B) = \sum_{A\subset B} m(A)$.

Conversely, if m is a function which satisfies (i) and (ii), then the function F defined

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by (iii) is a beleaf function on (Ω, J) [4]. We call m the mass function of F.

§2. Some examples of beleaf functions

We suggest here three kinds of beleaf functions which will be used to check some certain facts about conditional beleaf functions.

2.1 Let $(\Omega, 2^\Omega, P)$ be a finite probability space. Then for any $r \geq 1, F = P^r$ is a beleaf function.

Proof First we suppose $r = n$ be a integer, and $\Omega = \{x_1, \cdots, x_m\}$. Then from $P(x_1) + \cdots + P(x_m) = P(\Omega) = 1$ we get

$$[P(x_1) + \cdots + P(x_m)]^n$$
$$= \sum_{\{i_1,\cdots,i_k\}\subset\{1,\cdots,m\}} \sum_{\substack{0<j_{i_1},\cdots,j_{i_k}\leq n \\ j_{i_1}+\cdots+j_{i_k}=n}} \frac{n!}{j_{i_1}!\cdots j_{i_k}!} P^{j_{i_1}}(x_{i_1}) \cdots P^{k_{i_k}}(x_{i_k}).$$

Now define $m : 2^\Omega \to [0,1]$, by $m(\emptyset) = 0$,

$$m(x_{i_1}, \cdots, x_{i_k}) = \begin{cases} \displaystyle\sum_{\substack{0<j_{i_1},\cdots,j_{i_k}\leq n \\ j_{i_1}+\cdots+j_{i_k}=n}} \frac{n!}{j_{i_1}!\cdots j_{i_k}!} P^{j_{i_1}}(x_{i_1}) \cdots P^{k_{i_k}}(x_{i_k}), & k \leq n \\ 0, & k > n. \end{cases}$$

It is easy to see that $\sum\limits_{A\subset\Omega} m(A) = 1$. Then for any $B \subset \Omega$, we define $F(B) = \sum\limits_{A\subset B} m(A)$, F is a beleaf function on $(\Omega, 2^\Omega)$, and it is easy to check that $F(B) = P^n(B)$ for all $B \subset \Omega$.

We next consider $r \geq 1$ be any real number. To prove $F(B) = P^r(B)$ is a beleaf function on $(\Omega, 2^\Omega)$, we need to show

$$B \subset \Omega \Rightarrow \sum_{A\subset B} (-1)^{|B-A|} P^r(A) \geq 0. \tag{2.1}$$

Since B is finite we can write $\{A : A \subset B\} = \{A_1, \cdots, A_t, A_{t+1}, \cdots, A_s\}$ with $|B - A_i|$ is even for $1 \leq i \leq t$, and $|B - A_i|$ is odd for $t+1 \leq i \leq s$. Then we can rewrite (2.1) as

$$P^r(A_1) + \cdots + P^r(A_t) \geq P^r(A_{t+1}) + \cdots + P^r(A_s).$$

Let $g(r) = P^r(A_1) + \cdots + P^r(A_t)$ and $h(r) = P^r(A_{t+1}) + \cdots + P^r(A_s)$. Since $g(1) = h(1)$ and $g(n) \geq h(n)$ for all integer n, and $g(r)$ and $h(r)$ are convex functions of r, we can conclude that $g(r) \geq h(r)$ for alll real numbers $r \geq 1$, which implies $F = P^r$ is a beleaf function.

2.2 Let $\Omega = \{x_1, \cdots, x_m\}$, and $g : \Omega \to [0,1]$ such that

$$\prod_{i=1}^{m} (1 + tg(x_i)) = 1 + t \text{ for some } t > 0.$$

Then we can define $m : 2^{\Omega} \to [0,1]$ by $m(\emptyset) = 0$ and $m(\{x_{i_1}, \cdots, x_{i_k}\}) = t^{k-1}\prod_{j=1}^{k} g(x_{i_j})$. Then

$$\sum_{A\subset\Omega} m(A) = \sum_{\{i_1,\cdots,i_k\}\subset\{1,\cdots,m\}} t^{k-1}\prod_{j=1}^{k} g(x_{i_j}) = \frac{1}{t}[\prod_{i=1}^{n}(1+tg(x_i)) - 1] = 1.$$

So $F(B) = \sum_{A\subset B} m(A)$ is a beleaf function on $(\Omega, 2^{\Omega})$. For example $\Omega = \{x_1, x_2, x_3, x_4\}, g(x_1) = \frac{1}{2}, g(x_2) = \frac{1}{4}, g(x_3) = \frac{1}{16}, g(x_4) = \frac{1}{255}$ and $t = 1$, then $F(B) = \sum_{A\subset B} m(A) = \prod_{x_i\in B}(1+g(x_i)) - 1$ is a beleaf function.

2.3 Let $X = \{a, b, c\}, P$ be a probability measure on $(X, 2^X)$ corresponding to the random choice $P(a) = P(b) = P(c) = \frac{1}{3}$. Let $\Omega = \{(a,b),(a,c),(b,c),(c,b)\}$ and $\Gamma : X \to 2^{\Omega}$ given by $\Gamma(a) = \{(a,b),(a,c)\}, \Gamma(b) = \{(b,c)\}, \Gamma(c) = \{(c,b)\}$. We define $F : 2^{\Omega} \to [0,1]$ by $F(A) = P\{x : \Gamma(x) \subset A\}$ then $(\Omega, 2^{\Omega}, F)$ is a beleaf space with source $(X, 2^X, P, \Gamma)$.

§3. The criteria for beleaf conditioning

Let (Ω, J, F) be a beleaf space. Given a conditional operator $F(\cdot/\cdot)$ we wish it satisfies the following properties which are reasonable by taking a serious consideration.

(I) If F is an additive probability measure, $F(\cdot/\cdot)$ is as the same as probability conditioning.

(II) For any B with $F(B) \neq 0, F(\cdot/B)$ remains a beleaf function on (Ω, J).

(III) $F(\cdot/\cdot)$ is commutative, i.e. $F_B(A/C) = F_C(A/B)$, where $F_B(\cdot) = F(\cdot/B), F_C(\cdot) = F(\cdot/C)$.

(IV) $F(\cdot/\cdot)$ satisfies the sand which principle, i.e. $F(A) \geq \min(F(A/B), F(A/\bar{B}))$, where $\bar{B}$ is the complement of B.

The following conditional beleaf operators were proposed by some authors, none of which satisfies all the conditions above.

(1) $F^s(A/B) = F(A\cap B)/F(B)$ [1];
(2) $F^d(A/B) = [F(A\cup\bar{B}) - F(\bar{B})]/[1 - F(\bar{B})]$ [2];
(3) $F^w(A/B) = [F(A) - F(A\cap\bar{B})]/[1 - F(\bar{B})]$ [1];
(4) $F^h(A/B) = F(A\cap B)/[F(A\cap B) + 1 - F(A\cup\bar{B})]$ [3].

Proposition 3.1 *$F^s(\cdot/\cdot), F^d(\cdot/\cdot)$ and $F^w(\cdot/\cdot)$ satisfy all the conditions but (IV).*

Proof (I)It is easy to check.
(II) See [1] for $F^s(\cdot/\cdot)$ and $F^w(\cdot/\cdot)$, and see [2] for $F^d(\cdot/\cdot)$.

(III) Since

$$\begin{aligned}F_B^s(A/C) &= F_B^s(A\cap C)/F_B^s(C) = \{F(A\cap C\cap B)/F(B)\}/\{F(B\cap C)/F(B)\}\\ &= F(A\cap C\cap B)/F(B\cap C),\end{aligned}$$

$$\begin{aligned}F_B^d(A/C) &= \{F_B^d(A\cup\bar C - F_B^d(\bar C)\}/\{1-F_B^d(\bar C)\}\\ &= \{F(A\cup\bar C\cup\bar B)-F(\bar B))/(1-F(\bar B)) - (F(\bar C\cup\bar B)-F(\bar B))/(1-F(\bar B))\}\\ &\quad /\{1-(F(\bar C\cup\bar B)-F(\bar B))/(1-F(\bar B))\}\\ &= \{F(A\cup\bar B\cup\bar C)-F(\bar B\cup\bar C)\}/\{1-F(\bar B\cup\bar C)\},\\ F_B^w(A/C) &= \{F_B^w(A)-F_B^w(A\cap\bar C)\}/(1-F_B^w(\bar C))\\ &= \{(F(A)-F(A\cap\bar B))/(1-F(\bar B)) - (F(A\cap\bar C)-F(A\cap\bar C\cap\bar B))\}\\ &\quad /\{1-(F(\bar C)-F(\bar B\cap\bar C))/(1-F(\bar B))\}\\ &= \{F(A)-F(A\cap\bar B)-F(A\cap\bar C)+F(A\cap\bar C\cap\bar B)\}\\ &\quad /\{1-F(\bar B)-F(\bar C)+F(\bar B\cap\bar C)\}.\end{aligned}$$

Then by the symmetry of B and C, it is easy to see that $F_B^s(A/C) = F_C^s(A/B)$, $F_B^d(A/C) = D_C^d(A/B)$, and $F_B^w(A/C) = F_C^w(A/B)$.

(IV) Let $(\Omega, 2^\Omega, F)$ be the beleaf space given in 2.3. First we choose $A = \{(a,b),(b,c),(c,b)\}$, and $B=\{(a,c),(b,c)\}$, then

$$\begin{aligned}F(A) &= P\{x:\Gamma(x)\subset A\} = P(\{b,c\}) = 2/3,\\ F^s(A/B) &= F(A\cap B))/F(B) = P\{x:\Gamma(x)\subset A\cap B\}/P\{x:\Gamma(x)\subset B\}\\ &= P(\{b\})/P(\{b\}) = 1,\\ F^s(A/\bar B) &= F(A\cap\bar B)/F(\bar B) = P(\{c\})/P(\{c\}) = 1.\end{aligned}$$

So $F(A) < \min(F^s(A/B), F^s(A/\bar B))$.

Second we choose $A=\{(a,b),(a,c)\}$, and $B=\{(a,b),(c,b)\}$, then

$$\begin{aligned}F(A) &= P\{x:\Gamma(x)\subset A\} = P(\{b\}) = 1/3,\\ F^d(A/B) &= [F(A\cup\bar B)-F(\bar B)]/[1-F(\bar B)] = [P(\{a,b\})-P(\{b\})]/[1-P(\{b\})]\\ &= (2/3-1/3)/(1-1/3) = 1/2,\\ F^d(A/\bar B) &= [F(A\cup B)-F(B)]/[1-F(B)] = [P(\{b,c\})-P(\{c\})]/[1-P(\{c\})]\\ &= (2/3-1/3)/(1-1/3) = 1/2,\\ F^w(A/B) &= [F(A)-F(A\cap\bar B)]/[1-F(\bar B)] = [P(\{a\})-P(\emptyset)]/[1-P(\{b\})\\ &= (1/3)/(1-1/3) = 1/2,\\ F^w(A/\bar B) &= [F(A)-F(A\cap B)]/[1-F(B)] = [P(\{a\})-P(\emptyset)]/[1-P(\{c\})\\ &= (1/3)/(1-1/3) = 1/2.\end{aligned}$$

So

$$F(A) < \min(F^d(A/B), F^d(A/\bar B)), F(A) < \min(F^w(A/B), F^w(A/\bar B)).$$

Lemma 3.1 *Let (Ω, J, F) be a beleaf space, and $P = \{p : p$ is a probability on (Ω, J) with $p(A) \geq F(A)$ for all $A \in J\}$. If $F(\cdot/\cdot)$ satisfies $F(A/B) \leq inf_{p\in P}p(A/B)$, then $F(A) \geq min(F(A/B), F(A/\bar{B}))$.*

Proof Since $F(A) = \inf_{p\in P} p(A)$ [5], and for any $p^* \in P$

$$\begin{aligned} p^*(A) &\geq \min(p^*(A/B), p^*(A/\bar{B})) \\ &\geq \min(\inf_{p\in P} p(A/B), \inf_{p\in P} p(A/\bar{B})) \geq \min(F(A/B), F(A/\bar{B})). \end{aligned}$$

Hence

$$F(A) \geq \min(F(A/B), F(A/\bar{B})).$$

Proposition 3.2 *$F^h(\cdot/\cdot)$ satisfies the conditions (I)-(IV) but (III).*

Proof (I)It is easy to check. (II)See [3].

(IV) Let P be the set given in Lemma 3.1. Then for any $p \in P$

$$1 - p(A \cup \bar{B}) = 1 - p(\overline{(\bar{A} \cap B)}) = p(\bar{A} \cap B).$$

Now we have

$$\begin{aligned} F^h(A/B) &= F(A \cap B)/[F(A \cap B) + 1 - F(A \cup \bar{B})] \\ &= 1/[1 + (1 - F(A \cup \bar{B}))/F(A \cap B)] \leq 1/[1 + (1 - p(A \cup \bar{B})/p(A \cap B)] \\ &= p(A \cap B)/[p(A \cap B) + 1 - p(A \cup \bar{B})] = p(A \cap B)/[p(A \cap B) + p(\bar{A} \cap B)] \\ &= p(A \cap B)/p(B) = p(A/B). \end{aligned}$$

Then by Lemma 3.1 $F(A) \geq \min(F^h(A/B), F^h(A/\bar{B}))$.

(III) Let $(\Omega, 2^\Omega, F)$ be a beleaf space given in 2.2 with $\Omega = \{x_1, x_2, x_3, x_4\}$ and $g(x_1) = 1/2, g(x_2) = 1/4, g(x_3) = 1/16, g(x_4) = 1/255$ and $t = 1$. Now choose $A = \{x_1\}, B = \{x_1, x_2, x_3\}$ and $C = \{x_1, x_2\}$. Then

$$\begin{aligned} F^h_B(A/C) &= F^h_B(A \cap C)/\{F^h_B(A \cap C) + 1 - F^h_B(A \cup \bar{C})\} \\ &= \{F(A \cap C \cap B)/(F(A \cap C \cap B) + 1 - F((A \cap C) \cup \bar{B}))\} \\ &\quad /\{F(A \cap C \cap B)/(F(A \cap C \cap B) + 1 - F((A \cap C) \cup \bar{B})) + 1 \\ &\quad - F((A \cup \bar{C}) \cap B)/(F((A \cup \bar{C}) \cap B) + 1 - F(A \cup \bar{C} \cup \bar{B}))\} \\ &= 13515/27019, \\ F^h_C(A/B) &= F^h_C(A \cap B)/[F^h_C(A \cap B) + 1 - F^h_C(A \cup \bar{B})] \\ &= \}F(A \cap B \cap C)/(F(A \cap B \cap C) + 1 - F((A \cap B) \cup \bar{C}))\} \\ &\quad /\{F(A \cap B \cap C)/(F(A \cap B \cap C) + 1 - F((A \cap B) \cup \bar{C})) + 1 \\ &\quad - F((A \cup \bar{B}) \cap C)/(F((A \cup \bar{B}) \cap C) + 1 - F(A \cup \bar{B} \cup \bar{C}))\} \\ &= 5/9. \end{aligned}$$

So $F^h_B(A/C) \neq F^h_C(A/B)$.

§4. A desirable conditional beleaf operator

Lemma 4.1 *For any beleaf space (Ω, J, F) we have*

$$F(A) \geq min[min(F^s(A/B), F^s(A/\bar{B})), min(F^w(A/B), F^w(A/\bar{B}))].$$

Proof Suppose $F(A) \leq F^s(A/B) = F(A \cap B)/F(B)$, then $F(A \cap B) \geq F(A)F(B)$. It follows that

$$F^w(A/\bar{B}) = [F(A) - F(A \cap B)]/[1 - F(B)] \leq [F(A) - F(A)F(B)]/[1 - F(B)] = F(A).$$

Now the result follows.

From Lemma 4.1 one will guess that if $F(\cdot/\cdot)$ is the operator which we wish to find out, the value of $\min(F(A/B), F(A/\bar{B}))$ may lies between $\min(F^s(A/B), F^s(A/\bar{B}))$ and $\min(F^w(A/B), F^w(A/\bar{B}))$. So it is reasonable to consider $F(A/B)$ lise between $F^s(A/B)$ and $F^w(A/\bar{B})$. Based on this fact we suggest the following

$$F^n(A/B) = [F(A \cap B) + F(A) - F(A \cap \bar{B})]/[F(B) + 1 - F(\bar{B})].$$

Theorem $F^n(\cdot/\cdot)$ *satisfies all the conditions of (I)- (IV).*

Proof (I)It is easy to check.

(II) We need to prove $F^n(\cdot/B)$ satisfies $(*)$ and $(**)$.

$(*)$ Straightforward. $(**)$ Let I be a finite subset of N and $\{A\} \cup \{A_i : i \in I\} \subset J$ with $A \supset \cup_{i \in I} A_i$. Since $F^s(\cdot/B)$ and $F^w(\cdot/B)$ are beleaf functions on (Ω, J) we have

$$F(A \bigcap B) \geq \sum_{\emptyset \neq J \subset I} (-1)^{|J|+1} F((\bigcap_{j \in J} A_j) \bigcap B)$$

and

$$F(A) - F(A \bigcap \bar{B}) \geq \sum_{\emptyset \neq J \subset I} (-1)^{|J|+1} (F(\bigcap_{j \in J} A_j) - F((\bigcap_{j \in J} A_j) \bigcap \bar{B})).$$

Then it follows

$$\begin{aligned} F(A \bigcap B) + F(A) - F(A \bigcap \bar{B}) \quad \geq \quad & \sum_{\emptyset \neq J \subset I} (-1)^{|J|+1} (F(\bigcap_{j \in J} A_j) + F((\bigcap_{j \in J} A_j) \bigcap B) \\ & - F((\bigcap_{j \in J} A_j) \bigcap \bar{B})). \end{aligned}$$

Deviding both sides by $1 + F(B) - F(\bar{B})$ we get

$$F^n(A/B) \geq \sum_{\emptyset \neq J \subset I} (-1)^{|J|+1} F^n((\bigcap_{j \in J} A_j)/B).$$

Hence $F^n(\cdot/B)$ satisfies $(**)$.

(III) From

$$F_B^n(A/C) = [F_B^n(A \cap C) + F_B^n(A) - F_B^n(A \cap \bar{C})]/[1 + F_B^n(C) - F_B^n(\bar{C})]$$

and

$$\begin{aligned}
F_B^n(A \cap C) &= [F(A \cap C \cap B) + F(A \cap C) - F(A \cap C \cap \bar{B})]/[1 + F(B) - F(\bar{B})], \\
F_B^n(A) &= [F(A \cap B) + F(A) - F(A \cap \bar{B})]/[1 + F(B) - F(\bar{B})], \\
F_B^n(A \cap \bar{C}) &= [F(A \cap \bar{C} \cap B) + F(A \cap \bar{C}) - F(A \cap \bar{C} \cap \bar{B})]/[1 + F(B) - F(\bar{B})], \\
F_B^n(C) &= [F(B \cap C) + F(C) - F(\bar{B} \cap C)]/[1 + F(B) - F(\bar{B})], \\
F_B^n(\bar{C}) &= [F(\bar{C} \cap B) + F(\bar{C}) - F(\bar{C} \cap \bar{B})]/[1 + F(B) - F(\bar{B})],
\end{aligned}$$

we can get

$$\begin{aligned}
F_B^n(A/C) = \; & [F(A) + F(A \cap B) + F(A \cap C) - F(A \cap \bar{B}) - F(A \cap \bar{C}) + F(A \cap B \cap C) \\
& - F(A \cap \bar{B} \cap C) - F(A \cap B \cap \bar{C}) + F(A \cap \bar{B} \cap \bar{C})]/[1 + F(B) \\
& - F(\bar{B}) + F(C) - F(\bar{C}) + F(B \cap C) + F(\bar{B} \cap \bar{C}) - F(B \cap \bar{C}) - F(\bar{B} \cap C)].
\end{aligned}$$

Then by the symmetry of B and C it is easy to see that $F_B^n(A/C) = F_C^n(A/B)$.

(IV) If

$$F(A) < F^n(A/B) = [F(A) + F(A \cap B) - F(A \cap \bar{B})]/[1 + F(B) - F(\bar{B})], \tag{4.1}$$

$$F(A) < F^n(A/\bar{B}) = [F(A) + F(A \cap \bar{B}) - F(A \cap B)]/[1 + F(\bar{B}) - F(B)]. \tag{4.2}$$

hold at the same time, then since $1 + F(B) - F(\bar{B}) > 0$ and $1 + F(\bar{B}) - F(B) > 0$ it follows that

$$\begin{aligned}
& F(A)[1 + F(B) - F(\bar{B})] + F(A)[1 + F(\bar{B}) - F(B)] \\
& \quad < F(A) + F(A \cap B) - F(A \cap \bar{B}) + F(A) + F(A \cap \bar{B}) - F(A \cap B).
\end{aligned}$$

Then $2F(A) > 2F(A)$, which is impossible. So we have $F(A) \geq F^n(A/B)$ or $F(A) \geq F^n(A/\bar{B})$. Hence

$$F(A) \geq \min(F^n(A/B), F^n(A/\bar{B})).$$

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关于条件信任函数

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摘 要

本文首先基于理论和实践两方面的考虑, 给出了建立条件信任函数公式的准则; 然后依据给定的准则. 讨论了已有的几类条件信任函数公式的优点和弊病; 最后我们给出了一种新的条件信任函数公式, 其具备所期望的所有条件. 同时为验证条件信任函数的性质的需要, 我们还首次建立了几类有限集上信任函数的实例.