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Blow-up of Solutions of Semilinear Parabolic Equations*

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Abstract We study the existence of blow-up and the properties of set of blow-up points of solutions for some semilinear parabolic equations. We also discuss the single point blow-up of solution.

1. Introduction

We shall consider the following initial boundary value problem

$$u_t - u_{xx} = f(u, u_x, x, t), \qquad (x,t) \in (-a,a) \times (0,T) \tag{1}$$
$$u(\pm a, t) = 0, \qquad t \in [0,T) \tag{2}$$
$$u(x,0) = \phi(x), \qquad x \in [-a,a] \tag{3}$$

where f is a nonegative continuous differential function with respect to each variable, so is $\phi(x)$.

Recently there are many papers concerning the blow-up problem of the solutions to semilinear parabolic equations. For example [1]-[5]. In this paper, we shall discuss the blow-up problem for the more general form of equation.

2. The existence of blow-up

In this section, we consider the following initial boundary value problem for the special form of f.

$$u_t - u_{xx} = -(A(u))_x + B(u) + g(t), \quad (x,t) \in (-a,a) \times (o,T) \tag{4}$$
$$u(\pm a, t) = 0, \qquad t \in [0,T] \tag{5}$$
$$u(x,0) = \phi(x), \qquad x \in [-a,a] \tag{6}$$

where $\phi \in C^2, \phi(\pm a) = 0, \phi'' - (A(\phi))_x + B(\phi) > 0, \phi_r < 0 (r = |x|)$ and $A(u) \geq 0, A'(u) \geq 0, A''(u) \geq 0$ for $u \geq 0$ (for example, $A(u) = u^k, k \geq 2, B(u) = u^p, p > 2$), $g \in C, g(t) \geq 0$.

For sufficiently small T_0, there is a unique solution to problem (4)-(6). When $t \leq T_0$, by the maximum principle, we have $u > 0$ for all $(x,t) \in (-a,a) \times (0,T_0)$. If the global

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solution does not exist, there must exist a time T such that a positive solution exist when $0 < t < T$, and $u(x,t)$ tend to infinite when $t \to T$, namely

$$\lim_{t\to T} \sup_{x\in(-a,a)} u(x,t) = \infty. \tag{7}$$

Then we call the blow-up occurs to the solution $u(x,t)$, and time T is called blow-up time of the solution. We give the definition of blow-up point as follows.

Definition *A point $x \in (-a,a)$ is called a blow-up point if there exists a sequence (x_m, t_m) such that*

$$t_m \to T, x_m \to x \text{ and } u(x_m,t_m) \to \infty \text{ if } m \to \infty.$$

In the following, we shall discuss the conditions under which the blow-up of the solutions to problem (4)-(6) occurs.

Theorem 2.1 *In problem (4)-(6), if $\phi(x) > M$ and M is sufficiently large. Then their solution blow-up in finite time.*

Proof We consider another problem

$$\begin{aligned} v_t - v_{xx} &= B(v) + g(t), \ (x,t) \in (-b,b) \times (0,T) & (8)\\ v(\pm b, t) &= 0, \qquad t \in [0,T) & (9)\\ v(x,0) &= M, \qquad x \in [-b,b], b < a & (10) \end{aligned}$$

if M is sufficiently large, then there exists a time T_0 at which $v(x,t)$ blow-up, and we can prove from $v(x,t) = v(-x,t)$ (see [4]) that

$$v_x < 0 \text{ if } 0 < x < b, 0 < t < T_0.$$

Set

$$\max_{x\in(-b,b)} v(x,t) = v(0,t) = m(t), \tag{11}$$

$$r(x) = \int_0^t A'(m(\tau))\, d\tau, r_0 = r(t_0). \tag{12}$$

When estimating the blow-up rate of their solutions (as in [1]), we have

$$\int_0^{T_0} v(0,t)\, dt < \infty.$$

As in [4], set

$$\begin{aligned} \Omega_1 &= \{0 < x < b, 0 < t < T_0\},\\ \Omega_2 &= \{r(x) - r_0 < x < 0, 0 < t < T_0\},\\ \Omega_3 &= \{r(t) - r_0 - b < x < r(t) - r_0, 0 < t < T_0\} \end{aligned}$$

and introduce the function $w(x,t)$ as follows

$$\begin{aligned} w(x,t) &= v(x,t) \quad \text{in} \quad \Omega_1,\\ w(x,t) &= m(t) \quad \text{in} \quad \Omega_2,\\ w(x,t) &= v(x - r(t) + r_0, t) \quad \text{in} \quad \Omega_3. \end{aligned}$$

Then by the $v_x < 0$,

$$w_t - W_{xx} + (A(W))_x - B(W) - g(t) = A'(W)W_x = v_x A'(v) \le 0 \quad \text{in} \quad \Omega_1. \tag{13}$$

Next, since $-v_{xx}(0,t) \ge 0$, thus

$$\begin{aligned} & w_t - w_{xx} + (A(w))_x - B(w) - g(t) \\ & = w_t - w_{xx} + w_x A'(w) - B(w) - g(t) \le m'(t) - B(m(t)) - g(t) \\ & \le v_t(0,t) - v_{xx}(0,t) - B(v(0,t)) - g(t) = 0 \quad \text{in} \quad \Omega_2. \end{aligned} \tag{14}$$

Finally, we have

$$\begin{aligned} & w_t - w_{xx} + (A(w))_x - B(w) - g(t) = v_t - v_x r'(t) - v_{xx} + v_x A'(v) - B(v) - g(t) \\ = \; & v_x A'(v) - v_x r'(t) = v_x[A'(v) - A'(m(t))] \le 0 \quad \text{in} \quad \Omega_3 \end{aligned} \tag{15}$$

Since $v_x \ge 0$, if $x < 0$.

Also, w and w_x are continuous across $\partial\Omega_1 \cup \partial\Omega_2$ and $\partial\Omega_2 \cup \partial\Omega_3$, and w is the lower solution to problem (4) in the interior of $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$.

Since

$$\begin{aligned} & w(x,0) = M > 0, \quad -b' < x < b, b' = b + r_0, \\ & w(x,t) = 0, \quad (x,t) \in \partial_p\Omega \setminus t = 0, \end{aligned}$$

where $\partial_p\Omega$ is the parabolic boundary of Ω. Consequently, if $a > b', \phi(x) \ge M$, then $u(x,t) \ge w(x,t)$ when $-b' < x < b$, and $v(x,t)$ blow-up at a finite time when M is sufficiently large, therefore $w(x,t)$ blow-up too. As a result, $u(x,t)$ blow-up at a finite time. That ends the proof. □

Another sufficient condition for the occurrence of blow-up is given as follows.

Assume $\bar{x} \in (-a,a)$. Set $\rho = a - |\bar{x}|, B(\bar{x},\rho) = \{x : |x - \bar{x}| \le \rho\} \subset (-a,a)$.

Theorem 2.2 *For problems (4)-(6), if*

$$\int_0^T g(t)\,dt = \infty, \tag{16}$$

then for their solutions $u(x,t)$, we have

$$\lim_{t \to r^-} u(x,t) = +\infty \quad \textit{for all} \quad x \in B(\bar{x},\rho) \tag{17}$$

and $u(x,t)$ blow-up everywhere in $(-a,a)$.

Proof Suppose $\bar{x} \in (-a,a)$, on the $B(\bar{x},\rho)$, the solution $v(x,t)$ to problem

$$\begin{aligned} v_t - v_{xx} &= B(v) + g(t), \quad (x,t) \in (-a,a) \times (0,T) \\ v(\pm a, t) &= 0, \quad t \in [0,T) \\ v(x,0) &= 0, \quad x \in [-a,a] \end{aligned}$$

is an upper solution for problem

$$\begin{aligned} \tilde{v}_t - \tilde{v_{xx}} &= g(t), \quad (x,t) \in \{|x-\bar{x}| < \rho\} \times (0,T), & (18) \\ \tilde{v}(x,t) &= 0, \quad |x-\bar{x}| = \rho, t \in [0,T), & (19) \\ \tilde{v}(x,0) &= 0, \quad |x-\bar{x}| \le \rho. & (20) \end{aligned}$$

Meanwhile, similar to Theorem 2.1(now $M = 0$), we may prove $v(x,t)$ is a lower solution for (4)-(6). In view of $\int_0^T g(t)\,dt = +\infty$, thus $\tilde{v}(\bar{x},t) \to +\infty$, as $t \to T^-$ (cf. [5], Th. 4.1). And since $v(x,t) \ge \tilde{v}(x,t)$ on $|x-\bar{x}| < \rho, t \in (0,T)$, hence $v(\bar{x},t) \to \infty$, as $t \to T^-$. But $\bar{x} \in (-a,a)$ is arbitrary, thus $v(x,t)$ blow-up occurs everywhere, and so $u(x,t)$ blow-up, too. $\square$

3. Properties of blow-up set

We consider the properties of blow-up set of solutions to the following initial boundary problem

$$\begin{aligned} u_t - u_{xx} &= f(u, u_x, x, t), \quad (x,t) \in (-a,a) \times (0,T) & (21) \\ u(\pm a, t) &= 0, \quad t \in [0,T), & (22) \\ u(x,0) &= \phi(x), \quad x \in [-a,a]. & (23) \end{aligned}$$

Theorem 3.1 *Consider (21)-(23). Suppose*

(1). $f(u,p,r,t) \in C^1, f \ge 0, f_1 \ge 0, f(0,0,x,t) \ge 0$, *here and later f_i denotes the partial derivative of f with respect to the ith variable.*

(2). *There exists a positive function $F(u), F'(u) > 0, F''(u) > 0$ for $u \ge 0$. And $F(0) = 0$ if $f(0,0,x,t) = 0$, and assume $\int^\infty \frac{du}{F(u)} < \infty$. If there exists $\varepsilon > 0$ small enough such that*

$$f_1 F - fF' \ge 2\varepsilon F F' + \varepsilon(a - \alpha_0) f_2 F F', \tag{24}$$

where $a - \alpha_0$ is properly small.

(3). $f_3 < 0$, *if* $x > 0$; $f_3 > 0$, *if* $x < 0$; $\phi_x < 0$, *if* $x > 0$; $\phi_x > 0$, *if* $x < 0$.

Then the blow-up set of $u(x,t)$ is a compact subset of $(-a,a)$.

Proof We shall prove that there exists a $\delta_0 > 0$, such that there is no blow-up point in $(-a,a) \setminus (-a+\delta, a-\delta_0)$. For this purpose, consider first $x = a$, set $\Omega_\alpha^+ = (\alpha, a), a - \alpha$ is sufficiently small, $\Omega_\alpha^- = (2\alpha - a, \alpha)$.

Define the function

$$W(x,t) = u(x,t) - u(2\alpha - x, t), x \in \Omega_\alpha^-, 0 < t < T. \tag{25}$$

Then

$$
\begin{aligned}
& w_t - w_{xx} = f(u, u_x, x, t) - f(u(2\alpha - x, t) u_x(2\alpha - x, t), 2\alpha - x, t) \\
& = f(u(x,t), u_x(x,t), x, t) - f(u(2\alpha - x, t), u_x(x,t), x, t) + f(u(2\alpha - x, t), u_x(x,t), x, t) \\
& \ - f(u(2\alpha - x, t), u_x(2\alpha - x, t), x, t) + f(u(2\alpha - x, t), u_x(2\alpha - x, t), x, t) \\
& \ - f(u(2\alpha - x, t), u_x(2\alpha - x, t), 2\alpha - x, t) \\
& f_1 w + f_2 w_x + f_3(2x - 2\alpha) \\
& \triangleq Cw + Dw_x + 2f_3(x - \alpha)),
\end{aligned} \tag{26}
$$

where C and D are bounded functions for $x \in \Omega_\alpha^-, 0 < t < T$. Since $f_3 < 0$, if $x < \alpha$, we conclude that

$$w_t - w_{xx} - Cw - Dw_x > 0. \tag{27}$$

Next, consider the values of w on the parabolic boundary of Ω_α^-.

$$
\begin{aligned}
& w(\alpha, t) = u(\alpha, t) - u(2\alpha - \alpha, t) = 0, \\
& w(2\alpha - a, t) = u(2\alpha - a, t) - u(a, t) = u(2\alpha - a, t) > 0, \\
& w(x, t) = u(x, 0) - u(2\alpha - x, 0) > 0.
\end{aligned}
$$

Thus, by the maximum principle, we have

$$w(x,t) > 0, \quad \text{in} \ \ \Omega_\alpha^- \times (0, T). \tag{28}$$

Since

$$w_x = 2\frac{\partial u}{\partial x} < 0, \quad \text{on} \quad \{x = \alpha\}, \tag{29}$$

and α is arbitrary, we conclude by varying the value of α that there exists α_0, such that $\frac{\partial u}{\partial x} < 0$ when $(x,t) \in \Omega_{\alpha_0}^+ \times (0,T)$. In order to complete the proof, we introduce a function

$$J = u_x + \varepsilon(x - \alpha_0) F(u) \quad \text{in} \quad \Omega_{\alpha_0}^+ \times (0, T). \tag{30}$$

Then we compute, using (21)

$$
\begin{aligned}
J_t - J_{xx} &= \frac{\partial}{\partial x}(u_t - u_{xx}) + \varepsilon(x - \alpha_0) F'(u)(u_t - u_{xx}) - \varepsilon(x - \alpha_0) F''(u) u_x^2 - 2\varepsilon F'(u) u_x \\
&= f_1 u_x + f_2 u_{xx} + f_3 + \varepsilon(x - \alpha_0) F' f - \varepsilon(x - \alpha_0) F'' u_x^2 - 2\varepsilon F' u_x \\
&= f_1[J - \varepsilon(x - \alpha_0) F] + f_2[J_x - \varepsilon F(u) - \varepsilon(x - \alpha_0) F' u_x] + f_3 \\
&\quad + \varepsilon(x - \alpha_0) F' f - \varepsilon(x - \alpha_0) F'' u_x^2 - 2\varepsilon F' u_x \\
&= [f_1 - \varepsilon(x - \alpha_0) f_2 F' - 2\varepsilon F'] J + f_2 J_x + f_3 \\
&\quad - \varepsilon(x - x_0)[F f_1 - F' f - \varepsilon(x - \alpha_0) f_2 F F' - 2\varepsilon F F'] \\
&\quad - \varepsilon F f_2 - \varepsilon(x - \alpha_0) F'' u_x^2.
\end{aligned} \tag{31}
$$

Set $C = f_1 - \varepsilon(x - \alpha_0) F' f - 2\varepsilon F'$, $D = f_2$. Then we have

$$J_t - J_{xx} - CJ - DJ_x = f_3 - \varepsilon(x - \alpha_0)[F f_1 - F' f - 2\varepsilon F F' - \varepsilon(x - \alpha_0) f_2 F F'] \leq 0,$$

where C and D are bounded functions. By the maximum principle, we conclude that J can not take positive maximum in $\Omega_{\alpha_0}^+ \times (0,T)$.

When ε is sufficiently small, we conclude that $J(x,0) < 0$ since $\phi_x < 0$ on $[\alpha_0, a) \times \{t = 0\}$. Next, we have $J(\alpha_0, t) < 0$ on $\{x = \alpha_0\} \times (0,T)$ from $u_x < 0$. It remains to show that $J(a,t) < 0$ on $\{x = a\} \times (0,T)$.

Consider the following initial boundary value problem

$$v_t - v_{xx} = 0, \quad (x,t) \in (-a,a) \times (0,T), \tag{21'}$$

$$v(\pm a, t) = 0, \quad t \in [0,T), \tag{22'}$$

$$v(x,0) = \varphi(x), \quad x \in [-a,a]. \tag{23'}$$

Since $f(u, u_x, x, t) \geq 0$, hence $u(x,t) \geq v(x,t)$. Then we have

$$\frac{\partial u}{\partial x} \leq \frac{\partial v}{\partial x} \leq -c_0 < 0 \qquad \text{on} \ \ \{x = a\} \times (0,T). \tag{32}$$

Therefore, we have

$$J(a,t) = u_x(a,t) + \varepsilon(a - \alpha_0)F(0) < 0 \quad \text{on} \quad \{x = a\} \times (0,T)$$

provided that ε is small enough.

Applying the maximum principle, we conclude that

$$J(x,t) \leq 0, \ \ (x,t) \in \Omega_{\alpha_0}^+ \times (0,T).$$

Hence we have

$$-u_x = |u_x| \geq \varepsilon(x - \alpha_0)F(u). \tag{33}$$

Set

$$G(s) = \int_s^\infty \frac{ds}{F(s)}. \tag{34}$$

For (33), integrating with respect to x from x_1 to x_2. Where $\alpha_0 < x_1 < x_2 < a$, we get

$$-\int_{x_1}^{x_2} \frac{u_x}{F(u)}\,dx \geq \varepsilon \int_{x_1}^{x_2} (x - \alpha_0)\,dx \tag{35}$$

and we have from (34)

$$\frac{\partial G(u)}{\partial x} = -\frac{u_x}{F(u)}, \quad \frac{\partial G(u)}{\partial x} \geq \varepsilon(x - \alpha_0).$$

Hence

$$G(u(x_2,t)) - G(u(x_1,t)) \geq \frac{\varepsilon}{2}[(x_2 - \alpha_0)^2 - (x_1 - \alpha_0)^2]. \tag{36}$$

It follows that

$$G(u(x_2,t)) \geq \frac{\varepsilon}{2}[(x_2 - \alpha_0)^2 - (x_1 - \alpha_0)^2] > 0. \tag{37}$$

When $t \to T^-$, there must be $\limsup_{t\to T^-} u(x_2,t) < \infty$, otherwise if $\limsup_{t\to T^-} u(x_2,t) = \infty$, then we have

$$G(u(x_2,t)) \to 0, \quad \text{as} \quad t \to T^-.$$

This contradicts (37). Thus

$$\limsup_{(x,t)\to(x_2,T^-)} u(x,t) < \infty, \quad \alpha_0 < x_2 < a.$$

This shows that every point x which satisfies $\alpha_0 < x < a$ is not blow-up point. Finally, the set of blow-up is a compact subset of $(-a,a)$ because of the continuity of $u(x,t)$, this completes the proof. □

4. Single point blow-up problem

Consider the following IBVP

$$u_t - u_{xx} = f(u, u_x, x, t), \qquad (x,t) \in (-a,a) \times (0,T), \tag{38}$$

$$u(\pm a, t) = 0, \qquad t \in [0,T), \tag{39}$$

$$u(x,0) = \varphi(x), \qquad x \in [-a,a], \tag{40}$$

where T is the blow-up time of the solution to problem (38)-(40). Suppose

(I) $f \in C^1$, and $f(u,u_x,x,t) \geq 0$ for $u \geq 0$; $f(0,0,x,t) = 0$; $f_1 > 0$; $f_3 < 0$, if $x > 0$; $f_3 > 0$, if $x < 0$, $f_4 > 0$.

(II) $\varphi(\pm a) = 0, \varphi \geq 0, \varphi'' + f(\varphi, \varphi', x, 0) \geq 0$.

(III) There exists a positive function $F(u)$ such that

$$F'(u) \geq 0, F''(u) \geq 0, F(0) = 0 \quad \text{for} \quad u \geq 0; \quad \text{and} \quad \int^{\infty} \frac{du}{F(u)} < \infty,$$

and there exists $\delta > 0$ such that

$$f_1 F - fF' \geq 2\delta FF' + \delta(x - x_1) f_2 FF'.$$

Lemma 4.1 *Suppose that (I)-(III) hold and $\varphi(x)$ satisfies*

$$\begin{cases} \varphi'(x) > 0, & \textit{if} \quad -a < x < 0, \\ \varphi'(x) < 0, & \textit{if} \quad 0 < x < a, \end{cases} \tag{41}$$

then for the solution $u(x,t)$ to the problem (38)-(40), there exist two curves $x = S^{\pm}(t), t \in (0,T)$ such that

$$\begin{aligned} u_x(x,t) > 0, &\quad \text{if} \quad -a < x < S^-(t), \\ u_x(x,t) = 0, &\quad \text{if} \quad S^-(t) \leq x \leq S^+(t), \\ u_x(x,t) < 0, &\quad \text{if} \quad S^+(t) < x < a. \end{aligned}$$

Proof Differentiating (38) with respect to x, we have

$$u_{xt} - u_{xxx} = f_1 u_x + f_2 u_{xx} + f_3. \tag{42}$$

Set $\Omega_T = (-a, a) \times (0, T)$ and denote by Q^+ an element of the set $\{(x,t)|u_x(x,t) > 0\}$. We shall prove the parabolic boundary of Q^+ must intersect the parabolic boundary of Ω_T. Indeed otherwise we have $u_x = 0$ on the parabolic boundary of Q^+. Hence, by the maximum principle for (42) we have $u_x \equiv 0$ in Q^+ and this contradicts the definition of Q^+ since the intersection of the set $\{(x,t)|u_x(x,t) > 0\}$ with the parabolic boundary of Ω_T is a connected arc. Thus, $\{u_x > 0\}$ has one element Q^+ in Ω_T.

Similarly, denote by Q^- the component of the set $\{u_x < 0\}$ and we can prove that Q^- is connected. Moreover, we can prove that for any $0 < \sigma < T$, $\{t = \sigma\}$ intersects $Q^\pm$ in just one interval. Thus, there exists $x = s^-(t)$ which is one section of the boundary of Q^+, and there exists $x = s^+(t)$ which is one section of the boundary of Q^-.

In the following, set

$$s^- = \liminf_{t \to T^-} s^-(t), \quad s^+ = \limsup_{t \to T^-} s^+(t). \tag{43}$$

Lemma 4.2 *For the problem (38)-(40), if assumptions (I) and (II) hold, then a point $x \in (-a, a)$ is a blow-up point if and only if*

$$s^- \leq x \leq s^+.$$

Proof First we prove that if a point x is a blow-up point, then $x \in [s^-, s^+]$. Indeed, for any $\varepsilon > 0$, choose t_0 such that $T - t_0$ is sufficiently small, set $x_1 = s^+ + \varepsilon$, $R_\varepsilon = \{(x,t)|x \in (x_1, a), t \in (t_0, T)\}$, then $u_x < 0$ in R_ε. In a similar way to the proof of Theorem 3.1, set

$$J = u_x + \delta(x - x_1)F(u),$$

we can conclude that there is no blow-up point in $x_1 < x < a$, nor in $-a < x < x_0 (x_0 = s^- - \varepsilon)$. This means that if x is a blow-up point then $x \in [s^-, s^+]$.

Below we shall prove that if $x \in [s^-, s^+]$, then x is a blow-up point. In fact, there exists sequences $t_m \to T$ such that $x \in [s^-(t_m), s^+(t_m)]$, namely, $u(x, t_m) = \max_{-a \leq x \leq a} u(x, t_m)$, hence $u(x, t_m) \to \infty$, when $m \to \infty$, that is, x is a blow-up point. □

Theorem 4.3 *If the above assumptions (I)-(III) hold, then there is only one blow-up point of the solutions to problem (38)-(40).*

Proof By Lemma 4.1 and Lemma 4.2, we need only to show $s^+ = s^-$ is sufficiently small enough. Suppose $s^+ > s^-$ and let $0 < \varepsilon < \frac{s^+ - s^-}{2}$. Set $x_0 = s^- - \varepsilon$, then by Lemma 4.2, we have $u(x_0, t) \leq c$. When $0 \leq t < T$.

Now we choose $t_0 < T$ such that $T - t_0$ is sufficiently small and $s^+(t_0) > s^+ - \varepsilon$ and $u(x_1, t_0) > c$. Where $x_1 = s^+(t_0)$, then we have $u(x_1, t) > u(x_0, t)$ if $t_0 \leq t < T$. Assume

$\alpha = \frac{x_0+x_1}{2}$. Without loss of generality, we can assume $\alpha < 0$. Set $R = \{(x,t)|x_0 < x < \alpha, t_0 < t < T\}$.

Consider function

$$w(x,t) = u(x,t) - u(2\alpha - x, t), \quad (x,t) \in R. \tag{44}$$

Then w satisfies

$$\begin{aligned} w_t - w_{xx} &= f(u(x,t), u_x(x,t), x, t) - f(u(2\alpha - x,t), u_x(2\alpha - x, t), 2\alpha - x, t) \\ &= f_1 w + f_2 w_x + f_3(2x - 2\alpha), \end{aligned}$$

namely,

$$w_t - w_{xx} - f_1 w - f_2 w_x = 2(x-\alpha) f_3 < 0. \tag{45}$$

Since $f_3 > 0$ in R.

On the parabolic boundary of R, we have

$$\begin{aligned} &w(\alpha,t) = u(\alpha,t) - u(\alpha,t) = 0, \\ &w(x_0,t) = u(x_0,t) - u(x_1,t) < 0, \\ &w(x,t_0) = u(x,t_0) - u(2\alpha - x, t_0) \le 0, \end{aligned}$$

since $u_x(x,t_0) \ge 0$ and $u_x(x,t_0) \not\equiv 0$, if $x \in (x_0, x_1)$.

By the maximum principle, we can conclude

$$u(x,t) < u(2\alpha - x, t) \quad \text{in} \quad R.$$

Hence for any $t \in (t_0, T)$, the maximum of $u(x,t)$ for $x_0 \le x \le x_1$ cannot be attained for $x \in [x_0, \alpha)$. Thus we have $s^-(t) \ge \alpha$, when $t_0 < t < T$. Consequently, $s^- \ge \alpha$, namely

$$s^- \ge \alpha = \frac{x_0 + x_1}{2} = \frac{s^+(t_0) + s^- - \varepsilon}{2} > \frac{s^+ - \varepsilon + s^- - \varepsilon}{2} = \frac{s^+ + s^-}{2} - \varepsilon.$$

Therefore

$$\varepsilon > \frac{s^+ + s^-}{2} - s^- = \frac{s^+ - s^-}{2}.$$

This contradicts the above assumption. This shows $s^+ = s^-$. The proof is complete. □

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半线性抛物型方程解的爆破

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摘　要

在本文中我们研究了半线性抛物型方程解的 Blow－up 的存在性和 Blow－up 点集的性质，也讨论了单点 Blow－up 和有关 Blow－up 率问题.