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Singulairty and Overconvergence of General Laguerre Series*

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By $L_n^{(\alpha)}(z)$ denote the general Laguerre polynomial [2, p.97]

$$\frac{1}{n!}e^z z^{-\alpha}(\frac{d}{dz})^n(e^{-z}z^{n+\alpha}),\quad \alpha > -1$$

and by E_τ denote the parabola: $\mathrm{Re}\sqrt{-z} = \tau \ \ (\tau > 0)$, i.e., $z = -(\tau + iy)^2, -\infty < y < +\infty$. Hereafter $\sqrt{-z} > 0$ for $z < 0$.

For a general Laguerre series:

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z), \alpha > -1, \tag{1}$$

it is well known that if

$$\tau = -\limsup(2n^{\frac{1}{2}})^{-1}\log|a_n|, 0 < \tau < +\infty, \tag{2}$$

then E_τ is its parabola of convergence [1, p.621–622].

1. Singularity

At first we give the following definition which play an important role in Theorem 1.

Definition *Let $\sum_{n=0}^{\infty} t_n$ be a complex series. Suppose that there is a closed angular region D of opening $< \pi$ (i.e., $D:\ \beta_1 \le Argz \le \beta_2, \beta_2 - \beta_1 < \pi$) such that the sequence $\{T_n\}$ of its partial sums satisfies the following conditions:*

$$(1)\quad T_n \to \infty, n \to \infty; \qquad (2)\quad T_n \in D, n \ge N,$$

then the series $\sum_{n=0}^{\infty} t_n$ is said to be general properly divergent.

This definition is a generalization for the concept of classical properly divergent [3, p.215].

Theorem 1 *Let E_1 be the parabola of convergence of the series (1). If the series (1) is general properly divergent at a point $z_0 \in E_1$, then this point z_0 is a singularity of the sum*

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function $f(z)$ of the series (1).

Proof Denote $S_n(z_0)=\sum_{k=0}^{n} a_k L_k^{(\alpha)}(z)$, $S_{-1}(z_0)=0$, we have

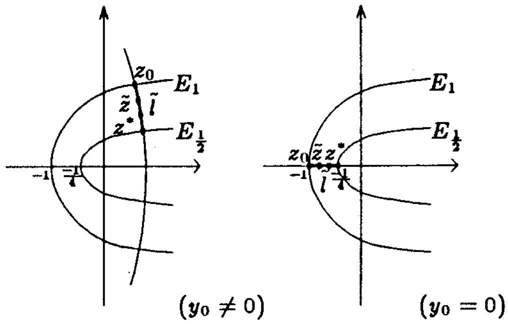
$$f(z)=\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z)=\sum_{n=0}^{\infty}(S_n(z_0)-S_{n-1}(z_0)\frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(z_0)},\quad z\in G_1, \tag{3}$$

hereafter G_τ : $\operatorname{Re}\sqrt{-z}<\tau$ and D_τ : $\operatorname{Re}\sqrt{-z}>\tau$.

Let $z_0=-(1+iy_0)^2\in E_1$. Take another parabola $\tilde{l}:\varsigma=-(t+iy_0)^2, \frac{1}{2}\le t\le 1$ through z_0. Now we prove that when $z\in\tilde{l}, z\neq z_0$, the series

$$\sum_{n=0}^{\infty} S_n(z_0)\frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(z_0)} \tag{4}$$

is convergent.



Take $z^*=-(\mu^*+iy_0)^2$ and $\tilde{z}=-(\tilde{\mu}+iy_0)^2, \frac{1}{2}\le\mu^*<\tilde{\mu}<1$. Since the series (1) is convergent at $\tilde{z}$, we have

$$|a_n L_n^{(\alpha)}(\tilde{z})|\le k. \tag{5}$$

Using the known asymptotic formula [2, p.193]

$$L_n^{(\alpha)}(z)=\frac{1}{2}\pi^{-\frac{1}{2}}e^{\frac{z}{2}}(-z)^{-\frac{\alpha}{2}-\frac{1}{4}}n^{\frac{\alpha}{2}-\frac{1}{4}}\exp\{2\sqrt{n}\sqrt{-z}\}\{1+O(\frac{1}{\sqrt{n}})\}, n>0, z\in\bar{D}_{\frac{1}{2}}, \tag{6}$$

we can get when z lies in every bounded closed region inside of $\bar{D}_{\frac{1}{2}}$,

$$K_1 n^{\frac{\alpha}{2}-\frac{1}{4}}e^{2\sqrt{n}\operatorname{Re}\sqrt{-z}}\le|L_n^{(\alpha)}(z)|\le K_2 n^{\frac{\alpha}{2}-\frac{1}{4}}e^{2\sqrt{n}\operatorname{Re}\sqrt{-z}}, n\ge M, \tag{7}$$

specially, when $z\in\tilde{l}$, the formula (7) is also valid. Noticing that $\tilde{z}, z_0, z^*\in\tilde{l}$, from (5) and $\operatorname{Re}\sqrt{-\tilde{z}}=\tilde{\mu}$ we get

$$|a_n|\le kn^{-\frac{\alpha}{2}+\frac{1}{4}}e^{-2\sqrt{n}\tilde{\mu}},\quad n\ge M, \tag{8}$$

furthermore from (7) and $\operatorname{Re}\sqrt{-z_0}=1, \operatorname{Re}\sqrt{-z^*}=\mu^*$ we get

$$|S_n(z_0)|\le k\sum_{k=M}^{n} e^{2\sqrt{k}(1-\tilde{\mu})}+\sum_{k=0}^{M-1}|a_k L_k^{(\alpha)}(z_0)|\le ke^{2\sqrt{n}(1-\mu_1)},\ \mu^*<\mu_1<\tilde{\mu}, n\ge M'\ge M$$

and

$$|S_n(z_0)\frac{L_n^{(\alpha)}(z^*)}{L_n^{(\alpha)}(z_0)}| \le ke^{2\sqrt{n}(\mu^*-\mu_1)}, \quad n \ge M',$$

so the series (4) is convergent at z^*.

Hence by (3), we get when $z \in \tilde{l}, z \ne z_0$,

$$f(z) = \sum_{n=0}^{\infty} S_n(z_0)\left(\frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(z_0)} - \frac{L_{n+1}^{(\alpha)}(z)}{L_{n+1}^{(\alpha)}(z_0)}\right). \tag{9}$$

Write $S_n(z_0) = R_n e^{i\theta_n}$. Because the series $\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z_0)$ is general properly divergent, for any given $G > 0$ there exists $\lambda, 0 < \epsilon_0 < \frac{\pi}{2}$ such that

$$R_n > G, \quad |\theta_n - \lambda| < \frac{\pi}{2} - \epsilon_0, \quad n \ge N_1. \tag{10}$$

Again write $G_n(\mu) = \frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(z_0)} - \frac{L_{n+1}^{(\alpha)}(z)}{L_{n+1}^{(\alpha)}(z_0)}$, here

$$z = -(\mu + iy_0)^2, \frac{1}{2} \le \mu < 1. \tag{11}$$

Applying the asymptotic formula (6), we know that $G_n(\mu) = A_n(\mu)B_n(\mu)$, where

$$A_n(\mu) = e^{\frac{1}{2}(1-\mu^2)}(e^{2\sqrt{n}(\mu-1)} - e^{2\sqrt{n+1}(\mu-1)}),$$

$$B_n(\mu) = e^{iy_0(1-\mu)}(\frac{\mu + iy_0}{1 + iy_0})^{-(\alpha+\frac{1}{2})}\left\{1 + O(\frac{1}{\sqrt{n}})\right\}. \tag{12}$$

Noticing that $\lim_{\substack{n\to\infty \\ \mu\to 1}} \mathrm{Re}\{B_n(\mu)\} = 1$, $\lim_{\substack{n\to\infty \\ \mu\to 1}} \mathrm{Im}\{B_n(\mu)\} = 0$ and $A_n(\mu) > 0$, we know that there exist μ_0 and $N \ge N_1$ such that

$$\frac{|\mathrm{Im}\{G_n(\mu)\}|}{\mathrm{Re}\{G_n(\mu)\}} < \frac{1}{4}\sin\epsilon_0, n \ge N, \frac{1}{2} \le \mu_0 < \mu < 1, \tag{13}$$

so by (9),

$$|f(-(\mu+iy)^2)| = |\sum_{n=0}^{\infty} R_n e^{i(\theta_n-\lambda)} G_n(\mu)| \ge \sum_{N}^{\infty} R_n \ \mathrm{Re}\{e^{i(\theta_n-\lambda)}G_n(\mu)\} - \sum_{0}^{N-1} R_n |G_n(\mu)|, \tag{14}$$

however by (10) and (13),

$$\begin{aligned}&\mathrm{Re}\{e^{i(\theta_n-\lambda)}G_n(\mu)\} = \cos(\theta_n - \lambda)\mathrm{Re}\{G_n(\mu)\} - \sin(\theta_n - \lambda)\mathrm{Im}\{G_n(\mu)\} \\ &> \frac{3}{4}\sin\epsilon_0 \ \mathrm{Re}\{G_n(\mu)\}, n \ge N, \frac{1}{2} \le \mu_0 < \mu < 1,\end{aligned}$$

so from (14) and $R_n > G$,

$$|f(-(\mu+iy_0)^2)| > \frac{3}{4}G\sin\epsilon_0 \mathrm{Re}\{\sum_{N}^{\infty} G_n(\mu)\} - \sum_{0}^{N-1} R_n|G_n(\mu)|,$$

again by (11) and (6), we get $\lim_{\mu\to 1}\sum_N^\infty G_n(\mu)=1, \lim_{\mu\to 1} G_n(\mu)=0$, hence $\lim_{\substack{z\to z_0\\ z\in l}} f(z)=\infty$, so Theorem 1 is proved.

Example Consider a series $\sum_1^\infty a_n L_n^{(\alpha)}(z)$, where $a_n = n^{-\frac{\alpha}{2}}(1+\frac{1}{n})^{-2n\sqrt{n}} e^{i(\frac{\pi}{4}-\frac{\pi}{8n})}$. By (2), its parabola of convergence is E_1. By (7), $|S_n(-1)| \geq \operatorname{Re}\{\sum_1^n a_k L_k^{(\alpha)}(-1)\} \geq k\cos\frac{\pi}{4}\sum_1^n k^{-\frac{1}{4}}$, so $S_n(-1)\to\infty, n\to\infty$. Write $S_n(-1)=R_n e^{i\theta_n}$. Clearly, $0<\theta_1=\frac{\pi}{8}<\frac{\pi}{4}$. If $0<\theta_{n-1}<\frac{\pi}{4}$, noticing that

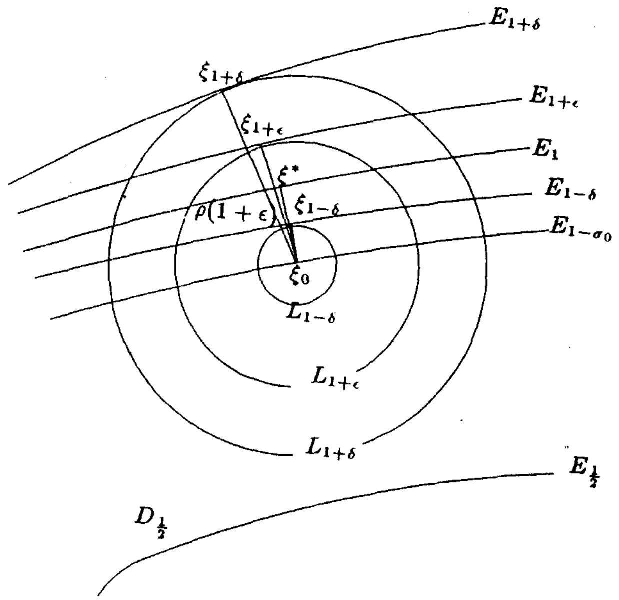
$$S_n(-1)=\{R_{n-1}\cos\theta_{n-1}+b_n\cos(\frac{\pi}{4}-\frac{\pi}{8n})\}+i\{R_{n-1}\sin\theta_{n-1}+b_n\sin(\frac{\pi}{4}-\frac{\pi}{8n})\},$$

where $b_n = n^{-\frac{\alpha}{2}}(1+\frac{1}{n})^{-2n\sqrt{n}} L_n^{(\alpha)}(-1)>0$. We get $\operatorname{Re}\{S_n(-1)\}>\operatorname{Im}\{S_n(-1)\}>0$, so $0<\theta_n<\frac{\pi}{4}$. Hence the series $\sum_1^\infty a_n L_n^{(\alpha)}(-1)$ is general properly divergent. By Theorem 1, the point $z=-1$ is a singularity.

2. Overconvergence

Theorem 2 *Let the parabola of convergence of the series (1) be E_1. If there are sequences of suffixes $p_k, q_k, (q_k \geq (1+\theta)p_k, \theta>0)$ such that $a_n=0$ $(p_k<n<q_k)$. Suppose that ξ^* is a regular point of sum function $f(z)$ on E_1. Then the sequence of partial sums $\{S_{p_k}(z)\}$ is convergent uniformly in a neighbourhood of ς^*.*

Proof Let $0<\epsilon<\delta<\sigma_0$. We consider four parabolas: $E_{1-\sigma_0}, E_{1-\delta}, E_{1+\epsilon}$ and $E_{1+\delta}$.



The normal line of E_1 through ξ^* intersects $E_{1-\sigma_0}$ at ξ_0. By $\rho(\tau)$ denote the distance from ξ_0 to E_τ, we draw three circles $L_{1-\delta}, L_{1+\epsilon}, L_{1+\delta}$ with contre ξ_0 and radii $\rho(1-\delta), \rho(1+\epsilon), \rho(1+\delta)$ respectively. Choose σ_0 so small that $L_{1+\delta}\subset D_{\frac{1}{2}}$, again choose δ so small that $f(z)$ is regular in and on $L_{1+\delta}$. Let

$$M_S = \max_{z \in L_s} |f(z) - S_{p_k}(z)|, S = 1-\delta, 1+\epsilon, 1+\delta. \tag{15}$$

Applying Hadamard's three-circles theorem,

$$M_{1+\epsilon}^{\log\frac{\rho(1+\delta)}{\rho(1-\delta)}} \le M_{1-\delta}^{\log\frac{\rho(1+\delta)}{\rho(1+\epsilon)}} \cdot M_{1+\delta}^{\log\frac{\rho(1+\epsilon)}{\rho(1-\delta)}}. \tag{16}$$

Since $L_{1-\delta} \subset \tilde{G}_{1-\delta} \subset G_1$ and $a_n = 0$ $(p_k < n < q_k)$, we have $M_{1-\delta} = \max_{z\in L_{1-\delta}} |\sum_{q_k}^{\infty} a_n L_n^{(\alpha)}(z)|$. Taking $\tilde{\mu} = 1-\eta$ $(0 < \eta < \frac{\delta}{2})$ in (8), we get from (7) and $\mathrm{Re}\sqrt{-z} \le 1-\delta$, $M_{1-\delta} \le k\sum_{q_k}^{\infty} e^{2\sqrt{n}(\eta-\delta)} \le ke^{2(\eta-\delta+\epsilon'_k)\sqrt{q_k}}$, here $\epsilon'_k = \frac{\ln\sqrt{q_k}}{2\sqrt{q_k}} \to 0$ $(k \to \infty)$.

Choose k so large that $\eta - \delta + \epsilon'_k < 0$, we have from $q_k \ge (1+\theta)p_k$,

$$M_{1-\delta} \le ke^{2(\eta-\delta+\epsilon'_k)(1+\theta)^{\frac{1}{2}}\sqrt{p_k}}.$$

On the other hand, from $L_0^{(\alpha)}(z) = 1$ [2, p.97] and $\mathrm{Re}\sqrt{-z} \le 1+\delta$, we can get

$$M_{1+\delta} \le \max_{z\in L_{1+\delta}} \{|f(z)| + \sum_{0}^{M-1} a_k L_k^{(\alpha)}(z)|\} + \max_{z\in L_{1+\delta}} |\sum_{M}^{p_k} a_n L_n^{(\alpha)}(z)| \le ke^{2(\eta+\delta+\epsilon''_k)\sqrt{p_k}},$$

here $\epsilon''_k = \frac{\ln\sqrt{p_k}}{2\sqrt{p_k}} \to 0$ $(k \to \infty)$ and M is stated as (8), so by (16) we get

$$M_{1+\epsilon}^{\log\frac{\rho(1+\delta)}{\rho(1-\delta)}} \le k\Omega_k^{\sqrt{p_k}}, \tag{17}$$

where $\Omega_k = \exp\{2(\eta-\delta+\epsilon'_k)(1+\theta)^{\frac{1}{2}}\log\frac{\rho(1+\delta)}{\rho(1+\epsilon)} + 2(\eta+\delta+\epsilon''_k)\log\frac{\rho(1+\epsilon)}{\rho(1-\delta)}\}$. Let $k \to \infty$. Then

$$\Omega_k \to \Omega = \exp\{2(\eta-\delta)(1+\theta)^{\frac{1}{2}}\log\frac{\rho(1+\delta)}{\rho(1+\epsilon)} + 2(\eta+\delta)\log\frac{\rho(1+\epsilon)}{\rho(1-\delta)}\}. \tag{18}$$

Again let $\epsilon \to 0, \eta \to 0$. We get from (18),

$$\Omega \to \tilde{\Omega} = \exp\{-2\delta(1+\theta)^{\frac{1}{2}}\log\frac{\rho(1+\delta)}{\rho(1)} + 2\delta\log\frac{\rho(1)}{\rho(1-\delta)}\}. \tag{19}$$

Since $\rho(\tau)$ is a radical function of τ,

$$\rho(1+\delta) = \rho(1) + \rho'(1)\delta + O(\delta^2),\ \rho(1-\delta) = \rho(1) - \rho'(1)\delta + O(\delta^2).$$

Furthermore $\log\frac{\rho(1+\delta)}{\rho(1)} = v\delta + O(\delta^2)$, $\log\frac{\rho(1)}{\rho(1-\delta)} = v\delta + O(\delta^2)$, where $v = \frac{\rho'(1)}{\rho(1)}$. Noticing that $\rho(\tau)$ is a strictly increasing function, we know that $v > 0$. From this and (19),we have $\tilde{\Omega} = \exp\{-2\delta(\sqrt{1+\theta}-1)(v\delta + O(\delta^2))\}$, so we can choose δ so small that $0 < \tilde{\Omega} < 1$. Again by (19), choose ϵ, η so small that $0 < \Omega < 1$, noticing that (18): $\Omega_k \to \Omega, k \to \infty$ and $\sqrt{p_k} \to +\infty, k \to \infty$, we obtain $\Omega_k^{p_k} \to 0, k \to \infty$. By (17) and (15), $\max_{z\in L_{1+\epsilon}} |f(z) - S_{p_k}(z)| \to 0, k \to \infty$. Applying the maximum modules principle, the sequence $\{S_{p_k}(z)\}$ is convergent uniformly in $|z - \xi_0| \le \rho(1+\epsilon)$, but $|\xi^* - \xi_0| = \rho(1) < \rho(1+\epsilon)$. Theorem 2 is proved.

References

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广义 Laguerre 级数的奇点和过度收敛

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摘 要

本文在§1中推广了古典正规发散的概念,给出了广义 Laguerre 级数在收敛抛物线上的奇点的判断定理.在§2中给出了广义 Laguerre 级数的"Ostrowski"型的过度收敛定理.