Journal of Mathematical Research & Exposition
Vol.13, No.3, Aug. 1993

A Note on Perfect Rings*

Chen Miaosen
(Dept. of Math., Zhejiang Normal Univ., Jinhua)

Chase [1] proved that for a commutative ring R, the direct product of R over infinite set A, R^A is projective if and only if R is an Artinian ring. And there exists a counterexample showing that if R is not commutative, then the above result need not hold. Hence it is very natural to ask such a question: Over which ring R, projectivity of R^A assume that R is Artinian. In this short note, we discuss the situation when such a ring becomes an Artinian ring.

All rings here are assumed associative with identity and all modules are unitary.

Proposition 1 *Let R be a left perfect ring. If the Jacobson radical $J(R)$ of R is a finitely generated right(left) ideal of R, then R is a right(left) Artinian ring.*

Proof Assume that J is a finitely generated right ideal and $\{a_1,\cdots,a_n\}$ is a set of generators of J. Then for any $x,y\in J, x=\sum_{i=1}^n a_ir_i, y=\sum_{i=1}^n a_is_i$, we have $xy=\sum_{i,j=1}^n a_ir_ia_js_j$. Since $r_ia_js_j\in J$, so $r_ia_js_j=\sum_{k=1}^n a_ku_k$, hence $xy=\sum_{i,j=1}^n a_ia_ju_j$, that is, J^2 is a finitely generated right ideal of R. Same as above, we can prove that J^k is finitely generated for every k and the set $\{b_1,\cdots,b_k|b_i\in\{a_1,\cdots,a_n\}\}$ is a set of generators of J^k.

We assert that J is a nilpotent ideal of R. In fact, if it is not the case, for any k, there exist $b_1,\cdots,b_k\in\{a_1,\cdots,a_n\}$ such that $b_1\cdots b_k\neq 0$. Then we construct set $F_k=\{b_1\cdots b_k\neq 0|b_i\in\{a_1,\cdots,a_n\}\}$ and a map Φ_k from F_k to $\mathrm{Pow}(F_{k+1})$ as $\Phi_k(b_1\cdots b_k)=\{b_1\cdots b_kb_{k+1}\neq 0|b_{k+1}\in\{a,\cdots,a_n\}\}$. Thus $(\mathcal{G},\phi)$ has paths with any finite length (see [2, p.40, 41.10]). By König's graph lemma, we know that there exists sequence in $(\mathcal{G},\phi)$ with infinite length, that is, sequence $b_1,\cdots,b_k,\cdots$ with the property that $b_1\cdots b_k\neq 0$ for any k. But this contradicts to J is left T−nil. So J is nilpotent. We have a chain as follows

$$0=J^k\subseteq J^{k-1}\subseteq\cdots\subseteq J\subseteq R$$

with property J^i/J^{i+1} is finitely generated over ring R/J. So J^i/J^{i+1} is Artinian since R/J is semisimple Artinian. Hence R is right Artinian.

So far, we have proven, in fact, that if $\{a_1,\cdots,a_n\}\subseteq J$, then there exists a k such that $b_1\cdots b_k=0$ for $b_i\in\{a_1,\cdots,a_n\}$. If J is a finitely generated left ideal of R and $\{a_1,\cdots,a_n\}$ is a set of generators of J. From the above proof, we also know that J is a nilpotent ideal of R and then R is left Artinian.

*Received Sep. 6, 1991.

Definition *Let R be a ring. If for any $x \in R, xR = Rx$ (i.e., every onesided ideal of R is a twosided ideal of R), then R is called a Duo ring.*

Example A commutative(division) ring is, of course, a Duo ring. And there do exist some ring which is a Duo ring, but neither a commutative ring nor a division ring. For instance, a skew polynomial ring over a field is a non-trivial Duo ring.

Proposition 2 *Let R be a Duo ring with identity. If for any set A, R^A is a left projective module, then R is an Artinian ring.*

Proof By Chase Theorem [1], we know that R is a left perfect and right coherent. So there exist orthogonal idempotents $e_1, \cdots, e_n$ such that $1 = e_1 + \cdots + e_n$ and $e_i R e_i$ is a local ring. Thus

$$R = 1 \cdot R = \prod_{i=1}^{n} e_i R e_i.$$

It is obviously that $e_i R e_i$ is also a left perfect and right coherent ring. Then $e_i R e_i / J(e_i R e_i)$ is isomorphic to a direct summand of $\mathrm{Soc}(e_i R e_i)$. By [1, Theorem 3.3], we know that $e_i R e_i / J(e_i R e_i)$ is a finitely presented $e_i R e_i-$modula. By Schanuel Lemma [2, Lemma 11.28], we obtain that $J(e_i R e_i)$ is a finitely generated ideal of $e_i R e_i$, hence $e_i R e_i$ is Artinian by Proposition 1. Thus $R = \prod_{i=1}^{n} e_i R e_i$ is Artinian.

Although R is need not Artinain when R^A is left projective for any set A, we can show that R is a semiprimary ring as follows:

Theorem 3 *Let R be a ring such that for any set A. R^A is a left projective module. Then the left annihilators of R satisfying the ascending chain condition. Furthermore, R is a semiprimary ring.*

Proof If the left annihilators of R do not satisfy the ascending chain condition. Then there exists an infinite ascending chain of left annihilators of R

$$l_R(X_1) \subseteq l_R(X_2) \subseteq \cdots \subseteq l_R(X_n) \subseteq \cdots.$$

We take $0 \neq a_1 \in l_R(X_1)$ and $a_n \in l_R(X_n) \setminus l_R(X_{n-1})$ for all $n \geq 2$. Thus we have an infinite descending chain of right annihilators of R

$$r_R(a_1) \supseteq r_R(a_, a_2) \supseteq \cdots \supseteq r_R(a_1, \cdots, a_n) \supseteq \cdots.$$

From [1, Theorem 2.2], $r_R(a_1, \cdots, a_n)$ is finitely generated for every n. Since R is a left perfect ring, R satisfies the descending chain condition on finitely generated right ideals by Björk Theorem [2]. Thus we obtain a contradiction, so the left annihilators of R satisfy the ascending chain condition.

If $J = J(R)$ is not nilpotent, then there exists some n such that $l_R(J^n) = l_R(J^{n+1})$. Since $J^{n+1} \neq 0$, there exists some $x_1 \in J$ with $x_1 J^n \neq 0$, that is, $x_1 \notin l_R(J^n) = l_R(J^{n+1})$, hence $x_1 J^{n+1} \neq 0$. So we can obtain a sequence of elements $x_1, \cdots, x_n, \cdots$ in J such that $x_1 \cdots x_k \in J \setminus l_R(J^n)$. But this contradicts to J is a left $T-$nil ideal of R. Thus R is semiprimary ring.

Corollary 4 *Let R be a left perfect and right coherent ring and $J = J(R)$ be the Jacobson* *radical of R. If J/J^2 is a finitely generated right(left) $R-$module, then R is a righ(left) Artinian ring.*

Proof If J/J^2 is a finitely generated right $R-$module, then there exists an $R-$epimorphism $f: R^{(n)} \to J/J^2$. Thus we have the following commutative diagram since $R^{(n)}$ is projective:

$$\begin{tikzcd}
 & & & R^{(n)} \arrow[dl, dashed, "\bar{f}"'] \arrow[d, "f"] & \\
0 \arrow[r] & J^2 \arrow[r] & J \arrow[r, "\alpha"] & J/J^2 \arrow[r] & 0
\end{tikzcd}$$

Hence $\mathrm{im}\bar{f} + J^2 = J$. By Theorem 3, we know that J is nilponent, so J^2 is a small submodule of J by [4, Lemma 28.3]. It follows that $\mathrm{im}\bar{f} = J$, that is, J is finitely generated right ideal of R. Thus R is a right Artinian ring by Proposition 1. Same as above, we can show that R is left Artinian when J/J^2 is finitely generated left $R-$module.

References

[1] S.U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc., **97**(1960).

[2] C. Faith, *Algebra I: Rings, Modules and Categories*, Springer-Verlag, Berlin-Heidelberg-New York, 1981.

[3] I.E. Björk, *Rings satisfying a minimum condition on principal ideals*, J. Reine. Angew. Math., **236**(1969), 112–119.

[4] F.W. Anderson and K.R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, Berlin-Heidelberg-New York, 1974.

关于Perfect环的一个注记

陈 淼 森

(浙江师范大学数学系,金华 321004)

摘 要

对于交换环 R,Chase[1]证明:对任意集 A,若 R^A 是射影模,则 R 是一个 Artin 环. 而对非交换环,有例子说明,此结论不成立. 本文讨论了对什么环,当 R 是射影模时,R 是一个 Artin 环.