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Weak Convergence and Bootstrap of Bivariate Product Limit Estimators under the Bivariate Competing Risks Case*

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Abstract This paper considers estimation of the bivariate survival function under the bivariate competing risks case. We give an iid representation for the PL estimator which is iid mean zero process and the remainder term is of order $O((n^{-1}\log n)^{3/4})$ a.s., weak convergence of the process to a two-dimensional-time Gaussian process is shown. Similar results are obtained for the bootstrap version.

1. Introduction

Methods for analyzing bivariate censored data have been studied by many authers over the last few decades. Relatively little research has been devoted to the analysis of bivariate observations in the bivariate competing risks model. However, survival and reliability studies often involve observations on paired individuals subject to censoring, more generally, competing risks model [1,2]. Let $(\vec{X},\vec{Y})$ be a pair of nonnegative random vectors, where $\vec{X}=(X_1^0,\cdots,X_r^0)$ and $\vec{Y}=(Y_1^0,\cdots,Y_K^0)$. The variables X_i^0 and $Y_j^0, i=1,\ldots,r; j=1,\ldots,k$ are survival times or censored times, which are thought of as competing risks. In the bivariate competing risks model, the complete collection of random vectors $(\vec{X},\vec{Y})$ is not possible. Instead, only the age at death given by (X,Y), where $X=\min\{X_1^0,\cdots,X_r^0\}, Y=\min\{Y_1^0,\cdots,Y_k^0\}$ and the cause of death are observed. One seeks to estimate the marginal survival probability.

In this paper we consider PL-type estimators of the marginal survival functions under the bivariate competing risks case. It is shown that the bivariate PL-type estimator can be written as a sum of mean zero iid processes and the remainder term is of the order $O((n^{-1}\log n)^{3/4})$ a.s. uniformly on compact sets. Using the representation we establish weak convergence and derive the corresponding results for the bootstrap estimators.

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2. Bivariate Competing Risks Model and Definition of Estimator

Let $X_1^0,\cdots,X_r^0$ and $Y_1^0,\cdots,Y_k^0$ be competing risks of bivariate survival times respectively. $X=\min\{X_1^0,\cdots,X_r^0\}, Y=\min\{Y_1^0,\cdots,Y_k^0\}, R_1=\{1,\cdots,r\}, R_2=\{1,\cdots,k\}$. We assume that the elements of R_1 represent the subscript of risks in the X−system, and the elements of R_2 represent the subscript of risks in the Y-system. Let φ_i denote the collection of nonempty subsets of $R_i, i=1,2$.

For $I_i\in\varphi_i$, Let $\varphi_{I_i}=\{J_i\in\varphi_i, J_i\cap I_i\neq\emptyset\}$ and $\bar{\varphi}_{I_i}=\varphi_i-\varphi_{I_i}, i=1,2$. Failure pattern I_1 occurs in the X-system with life length X if $\xi(\vec{X})$ is I_1. Similarly, failure pattern I_2 occurs in the Y- system with life length Y if $\xi(\vec{Y})$ is I_2. Where

$$\xi(\vec{X})=\begin{cases} I_1, & \text{if } X=X_i^0 \text{ for each } i\in I_1 \text{ and } X\neq X_i^0 \text{ for each } i\notin I_1\\ \emptyset, & \text{otherwise}\end{cases}$$

and

$$\xi(\vec{Y})=\begin{cases} I_2, & \text{if } Y=Y_i^0 \text{ for each } i\in I_2 \text{ and } Y\neq Y_i^0 \text{ for each } i\notin I_2,\\ \emptyset, & \text{otherwise}\end{cases}$$

while $\vec{X}=(X_1^0,\cdots,X_r^0)$ and $\vec{Y}=(Y_1^0,\cdots,Y_k^0)$.

The complete collection of random variables $X_1^0,\ldots,X_r^0$ and $Y_1^0,\ldots,Y_k^0$ is not observed under the bivariate competing risks case. Instead, only four quantities are observed: the bivariate age at death given by (X,Y), where $X=\min\{X_1^0,\cdots,X_r^0\}, Y=\min\{Y_1^0,\cdots,Y_k^0\}$ and the bivariate cause of death, labeled $(\xi(\vec{X}),\xi(\vec{Y}))$, given by $I_1\in\varphi_1$ such that $\xi(\vec{X})=I_1$ and $I_2\in\varphi_2$ such that $\xi(\vec{Y})=I_2$. When death results from exactly a pair of the $r\cdot k$ pairs of possible causes, as is usually assumed, then $(\xi(\vec{X}),\xi(\vec{Y}))$ is the index (i,j) for which $(X,Y)=(X_i^0,Y_j^0)$. The biomedical researchers are interested in making inferences about unobservable quantities (viz., the random variables $X_1^0,\cdots,X_r^0$ and $Y_1^0,\cdots,Y_k^0$) by using data from observable quantities—in this case, a pair of lifetimes (X,Y) and a pair of causes of edath $(\xi(\vec{X}),\xi(\vec{Y}))$. In particular, they seek to estimate the marginal survival function corresponding to a pair of given causes (or a pair of combination of causes) operating alone without competition from the other causes. That is, they wish to estimate the $(2^r-1)(2^k-1)$ survival probabilities

$$S_{I_1,I_2}(x,y)=P(\min_{i\in I_1}\{X_i^0\}>x,\min_{j\in I_2}\{Y_j^0\}>y)\triangleq P(X_{I_1}>x,Y_{I_2}>y).$$

In analyzing competing risks data, we assume two of the following:

(A1) Let $X_1^0,\ldots,X_r^0$ and $Y_1^0,\ldots,Y_k^0$ be two mutually independent sets of random variables. The vectors $\{X_i^0/i\in I_1\}$ are independent of $\{Y_j^0/j\in\bar{I}_2\}$ and $\{X_i^0/i\in\bar{I}_1\}$ are independent of $\{Y_j^0/j\in I_2\}$ where $\bar{I}_i=R_i-I_i, i=1,2$.

(A2) The functions $S_{I_1,I_2}(x,y)$ and $S_{\bar{I}_1,\bar{I}_2}(x,y)$ are absolutely continuous with respect to Lebesgue measure on R^2.

In the bivariate competing risks model, one observes a sample: $(X_i,Y_i,\xi(\vec{X}_i),\xi(\vec{Y}_i)), i=1,\ldots,n$, where $X_i=\min\{X_{1i}^0,\cdots,X_{ri}^0\}, Y_i=\min\{Y_{1i}^0,\cdots,Y_{ki}^0\}$,

$$\xi(\vec{X}_i)=\begin{cases} I_{1i}, & \text{if } X_i=X_{li}^0 \text{ for each } l\in I_{1i} \text{ and } X_i\neq X_{li}^0 \text{ for each } l\notin I_{1i}\\ \emptyset, & \text{otherwise}\end{cases},$$

and

$$\xi(\vec{Y}_i) = \begin{cases} I_{2i}, & \text{if } Y_i = Y_{li}^0 \text{ for each } l \in I_{2i} \text{ and } Y_i \neq Y_{li}^0 \text{ for each } l \notin I_{2i} \\ \emptyset, & \text{otherwise} \end{cases}.$$

Let $H(x,y) = P(X > x, Y > y)$ denote the survival function of (X,Y). By assumption (A1),

$$\begin{aligned} H(x,y) &= P(X > x, Y > y) = P(X_{I_1} > x, X_{\bar{I}_1} > x, Y_{I_2} > y, Y_{\bar{I}_2} > y) \\ &= P(X_{I_1} > x, Y_{I_2} > y) P(X_{\bar{I}_1} > x, Y_{\bar{I}_2} > y) \\ &= S_{I_1,I_2}(x,y) S_{\bar{I}_1,\bar{I}_2}(x,y). \end{aligned}$$

Based on the observations $(X_i, Y_i, \xi(\vec{X}_i), \xi(\vec{Y}_i))$ one would like to estimate $S_{I_1,I_2}(x,y)$.

We shall estimate $S_{I_1,I_2}(x,y)$ based on the fact that $S_{I_1,I_2}(x,y) = S_{I_1,I_2}(x,0) S_{I_1,I_2}(y/x)$, where $S_{I_1,I_2}(y/x) = P(Y_{I_2} > y / X_{I_1} > x)$. Let

$$\begin{aligned} N(x,y) &= \sum_{i=1}^{n} I(X_i > x, Y_i > y), \\ \alpha_i(x,y) &= I(X_i \leq x, Y_i > y, \xi(\vec{X}_i) \in \varphi_{I_1}), i = 1, \ldots, n, \\ \beta_j(x,y) &= I(X_j > x, Y_j \leq y, \xi(\vec{Y}_i) \in \varphi_{I_2}), j = 1, \ldots, n, \end{aligned}$$

where $I(\cdot)$ is the indicator function. To estimate $S_{I_1,I_2}(x,0)$, project all points (X_i, Y_i) vertically onto the X-axis and ignore the Y_i values. Let $\hat{S}_{I_1,I_2}(x,0)$ be the one-dimensional PL-type estimator of $S_{I_1,I_2}(x,0)$ based on $(X_i, \xi(\vec{X}_i)), i = 1, \ldots, n$. That is,

$$\hat{S}_{I_1,I_2}(x,0) = \begin{cases} \prod_{i=1}^{n} \left(\frac{N(X_{(i)},0)}{N(X_{(i)},0)+1} \right)^{\alpha_i(x,0)} & \text{if } x \leq X_{(n)} \\ 0 & \text{otherwise} \end{cases},$$

where $X_{(i)}$ denotes the i-th ordered value of $\{X_1, \ldots, X_n\}, i = 1, \ldots, n$.

To estimate $S_{I_1,I_2}(y/x)$, project all points (X_i, Y_i) for which $X_i > x$ horizontally to the line $X = x$, and ignore the X_i values. Let $\hat{S}_{I_1,I_2}(y/x)$ be the one-dimensional PL-type estimator of $S_{I_1,I_2}(y/x)$ based on $(Y_i, \xi(\vec{Y}_i))$, for which $X_i > x$. That is,

$$\hat{S}_{I_1,I_2}(y/x) = \begin{cases} \prod_{j=1}^{n} \left(\frac{N(x,Y_{(j)})}{N(x,Y_{(j)})+1} \right)^{\beta_j(x,y)} & \text{if } y \leq Y_{(n)} \\ 0 & \text{otherwise} \end{cases},$$

where $Y_{(n)}(x) = \max_{1 \leq i \leq n} \{Y_i : X_i > x\}$ and $Y_{(j)}$ denotes the j-th ordered value of $\{Y_1, \cdots, Y_k\}$, $j = 1, \ldots, n$.

Our estimator of $S_{I_1,I_2}(x,y)$ is

$$\hat{S}_{I_1,I_2}(x,y) = \hat{S}_{I_1,I_2}(x,0) \hat{S}_{I_1,I_2}(y/x). \tag{1}$$

For the bivariate case, we shall consider the bootstrap method [3] of drawing random samples (with replacement) $(X_i^*, Y_i^*, \xi(\vec{X}_i^*), \xi(\vec{Y}_i^*)) : i = 1, \ldots, n$ from the population

$\Omega^* = \{(X_i, Y_i, \xi(\vec{X}_i), \xi(\vec{Y}_i)) : i = 1, \ldots, n\}$, giving each element in Ω^* equal chance $(1/n)$ at each draw. The bivariate PL-type estimator $\hat{S}^*_{I_1,I_2}(x,y)$ is then constructed as the $\hat{S}_{I_1,I_2}(x,y)$ but using the bootstrap sample $\{(X_i^*, Y_i^*, \xi(\vec{X}_i^*), \xi(\vec{Y}_i^*)) : i = 1, \ldots, n\}$ instead thus $\hat{S}^*_{I_1,I_2}(x,y) = \hat{S}^*_{I_1,I_2}(x,0)\hat{S}^*_{I_1,I_2}(y/x)$.

3. Asymptotic Representations

We shall adopt the notations of Section 2 for the bivariate competing risks model, and define

$$\begin{cases} H(y/x) & = P(Y > y/X > x) \\ H_1(y/x) & = P(Y > y, \xi(\vec{Y}) \in \varphi_{I_2}/X > x) \\ H_{1X}(x,y) & = P(X > x, Y > y, \xi(\vec{X}) \in \varphi_{I_1}) \\ H_{1Y}(x,y) & = P(X > x, Y > y, \xi(\vec{Y}) \in \varphi_{I_2}) \\ H_{11}(x,y) & = P(X > x, Y > y, \xi(\vec{X}) \in \varphi_{I_1}, \xi(\vec{Y}) \in \varphi_{I_2},). \end{cases} \tag{2}$$

For positive reals u, v, x, y, let

$$A(u, \xi(\vec{X}), x) = -[g(u \wedge x) + (H(u,0))^{-1} I(u \le x, \xi(\vec{X}) \in \varphi_{I_1})] \tag{3}$$

where $g(u) = \int_0^u [H(x,0)]^{-2} dH_{1X}(x,0)$, and

$$A_x(v, \xi(\vec{Y}), y) = -[g_x(v \wedge y) + (H(v/x))^{-1} I(v \le y, \xi(\vec{Y}) \in \varphi_{I_2})], \tag{4}$$

where $g_u(v) = \int_0^v [H(y/u)]^{-2} dH_1(y/u)$. Let (S,T) be any point with $H(S,T) > 0$, let

$$m_x = \sum_{i=1}^n I(X_i > x), \; m_x^* = \sum_{i=1}^n I(X_i^* > x).$$

Lemma 1

(a) $$\log \hat{S}_{I_1,I_2}(y/x) - \log S_{I_1,I_2}(y/x) = m_x^{-1} \sum_{i=1}^n A_x(Y_i, \xi(\vec{Y}_i), y) I(X_i > x) + R_n(y/x),$$

where $\sup_{0 \le x \le S, 0 \le y \le T} |R_n(y/x)| = O((n^{-1} \log n)^{3/4})$ *a.s.*

(b) $$\log \hat{S}^*_{I_1,I_2}(y/x) - \log \hat{S}_{I_1,I_2}(y/x) = {m^*}_x^{-1} \sum_{i=1}^n [A_x(Y_i^*, \xi(\vec{Y}_i^*), y) I(X_i^* > x)$$
$$- A_x(Y_i, \xi(\vec{Y}_i), y) I(X_i > x)] + R_n^*(y/x),$$

where $\sup_{0 \le x \le S, 0 \le y \le T} |R_n^*(y/x)| = O_{P^*}((n^{-1} \log n)^{3/4})$ *a.s., and* P^* *stands for the bootstrap probability.*

The proof is tedious and similar to [4].

Theorem 1

(a) $\log \hat{S}_{I_1,I_2}(x,y) - \log S_{I_1,I_2}(x,y) = n^{-1}\sum_{i=1}^{n}\eta(X_i, Y_i, \xi(\vec{X}_i), \xi(\vec{Y}_i), x, y) + R_n(x,y)$,

where $\sup_{0\le x\le S, 0\le y\le T}|R_n(x,y)| = O((n^{-1}\log n)^{3/4})$ *a.s., and*

$$\eta(u, v, \xi(\vec{X}_i), \xi(\vec{Y}_i), x, y) = A(u, \xi(\vec{X}), x) + [H(x,0)]^{-1}A_x(v, \xi(\vec{Y}), y)I(u > x).$$

(b) $$\hat{S}_{I_1,I_2}(x,y) - S_{I_1,I_2}(x,y)$$
$$= n^{-1}S_{I_1,I_2}(x,y)\sum_{i=1}^{n}\eta(X_i, Y_i, \xi(\vec{X}_i), \xi(\vec{Y}_i), x, y) + R'_n(x,y),$$

where $\sup_{0\le x\le S, 0\le y\le T}|R'_n(x,y)| = O((n^{-1}\log n)^{3/4})$ *a.s.,*

(c) $$\hat{S}_{I_1,I_2}(x,y) - S_{I_1,I_2}(x,y) = n^{-1}S_{I_1,I_2}(x,y)\sum_{i=1}^{n}[\eta(X_i^*, Y_i^*, \xi(\vec{X}_i^*), \xi(\vec{Y}_i^*), x, y)$$
$$-\eta(X_i, Y_i, \xi(\vec{X}_i), \xi(\vec{Y}_i), x, y)] + R_n^*(x,y),$$

where $\sup_{0\le x\le S, 0\le y\le T}|R_n^*(x,y)| = O_{p^*}((n^{-1}\log n)^{3/4})$ *a.s.*

Proof of (a) Theorem 1 of [5] and Lemma 1 imply that

$$\begin{aligned}
&\log \hat{S}_{I_1,I_2}(x,y) - \log S_{I_1,I_2}(x,y)\\
&\quad= [\log \hat{S}_{I_1,I_2}(x,0) - \log S_{I_1,I_2}(x,0)] + [\log \hat{S}_{I_1,I_2}(y/x) - \log S_{I_1,I_2}(y/x)]\\
&\quad= n^{-1}\sum_{i=1}^{n}A(X_i, \xi(\vec{X}_i), x) + m_x^{-1}\sum_{i=1}^{n}A_x(Y_i, \xi(\vec{Y}_i), y)I(X_i > x) + R_{n1}(x,y)\\
&\quad= n^{-1}\sum_{i=1}^{n}[\eta(X_i, Y_i, \xi(\vec{X}_i), \xi(\vec{Y}_i), x, y) + R_{n1}(x,y) + R_{n2}(x,y),
\end{aligned}$$

where $\sup_{0\le x\le S, 0\le y\le T}|R_{n1}(x,y)| = O((n^{-1}\log n)^{3/4})$ a.s., and

$$R_{n2}(x,y) = \{n/m_x - [H(x,0)]^{-1}\}n^{-1}\sum_{i=1}^{n}A_x(Y_i, \xi(\vec{Y}_i), y)I(X_i > x).$$

It is easy to see at this stage that, $R_{n2}(x,y) = O(n^{-1}\log\log n)$ a.s. for each (x,y). To prove that it holds uniformly for $0 \le x \le S, 0 \le y \le T$, we shall apply the functional LIL due to Theorem 4.1 of [6].

Let $Z_i = A_x(Y_i, \xi(\vec{Y}_i), y)I(X_i > x)$, Z_i takes values in $D[0,S] \times D[0,T]$ under the sup norm $\|\cdot\|$ on $[0,S] \times [0,T]$, and $S_n = \sum_{j=1}^{n} Z_j$, It is clear that $E\|Z_1\|^2 < \infty$, and hence condition (4.2) of [6] is satisfied. Condition (4.1) is satisfied due to the tightness of the process $n^{-1/2}S_n$ which is shown in Theorem 2. It then follows from Theorem 4.1 of [6] that $\|S_n/n\| = O((n^{-1}\log\log n)^{1/2})$ a.s. Also

$$\sup_{0\le x\le S} |n/m_x - [H(x,0)]^{-1}| = O((n^{-1}\log\log n)^{1/2}) \text{ a.s.},$$

from the LIL for empirical distribution and the fact that $H(S,0) > 0$. We have thus shown that $\sup_{0\le x\le S, 0\le y\le T} |R_{n2}(x,y)| = O(n^{-1}\log\log n)$ a.s. Part (a) now follows from $R_n(x,y) = R_{n1}(x,y) + R_{n2}(x,y)$.

Proof of (b) Let $Z_i = \eta(X_i, Y_i, \xi(\vec{X}_i), \xi(\vec{Y}_i), x, y)$. It can be checked easily that Z_i is uniformly bounded on $[0,S]\times[0,T]$. Applying Theorem 4.1 of [6] once again, we have

$$\sup_{0\le x\le S, 0\le y\le T} |n^{-1}\sum_{i=1}^{n} \eta(X_i, Y_i, \xi(\vec{X}_i), \xi(\vec{Y}_i), x, y)| = O((n^{-1}\log\log n)^{1/2}) \text{ a.s.}$$

(b) then follows from (a) and the two-term Taylor's expansion of

$$\hat{S}_{I_1,I_2}(x,y) - S_{I_1,I_2}(x,y) = \exp[\log \hat{S}_{I_1,I_2}(x,y)] - \exp[\log S_{I_1,I_2}(x,y)].$$

Proof of (c) Using Theorem 1 of [5] and Lemma 1 (b), the proof follows by mimicking the proofs of (a) and (b). □

The next LIL follows from the proof of Theorem 1 by applying Theorem 4.1 of [6].

Corollary 1 *Under the condition of Theorem 1,*

(a) $$\sup_{0\le x\le S, 0\le y\le T} |\hat{S}_{I_1,I_2}(x,y) - S_{I_1,I_2}(x,y)| = O((n^{-1}\log\log n)^{1/2}) \quad a.s.,$$

(b) $$\sup_{0\le x\le S, 0\le y\le T} |\hat{S}^*_{I_1,I_2}(x,y) - \hat{S}_{I_1,I_2}(x,y)| = O_{p^*}((n^{-1}\log\log n)^{1/2}) \quad a.s.$$

Let $\eta(x,y) = \eta(X, Y, \xi(\vec{X}), \xi(\vec{Y}), x, y)$ and $\Gamma((x,y),(x',y')) = \mathrm{Cov}(\eta(x,y), \eta(x',y'))$. The mean and covariance structure of the process $\{\eta(x,y)\}$ is given in the next proposition.

Proposition 1 *(a)* $E(\eta(x,y)) = 0$, *(b) Assume* $x \le x'$.

$$\Gamma((x,y),(x',y')) = -g(x) + \mathrm{II} + [H(x,0)]^{-1} \times \{\int_0^y H(v/x')[H(v/x)]^{-2} g_x(v\wedge y')dH_1(v/x)$$
$$- \int_0^y [H(v/x)]^{-1} g_x(v\wedge y')dH_1(v/x') - \int_0^{y\wedge y'} [H(v/x)H(v/x')]^{-1} dH_1(v/x')\},$$

where

$$\begin{aligned}\mathrm{II} \;=\; & [H(x,0)]^{-1}\{\int_{[x,\infty)\times(0,\infty)} g(u\wedge x')g_x(v\wedge y)dH(u,v) \\ & + \int_{[x,\infty)\times[0,y]} [g(u\wedge x')/H(v/x)]dH_{1Y}(u,v) \\ & + \int_{[x,x')\times[0,\infty)} [g_x(v\wedge y)/H(u,0)]dH_{1X}(u,v) \\ & + \int_{[x,x')\times[0,y]} [H(v/x)H(u,0)]^{-1} dH_{11}(u,v).\end{aligned}$$

The proof is tedious and similar to [5].

4. Weak Convergence

Let $\eta_i(x,y) = \eta(X_i, Y_i, \xi(\vec{X}_i), \xi(\vec{Y}_i), x, y)$, $\quad \bar{\eta}(x,y) = n^{-1}\sum_{i=1}^{n} \eta_i(x,y)$, $\quad \varsigma_i(x,y) = S_{I_1,I_2}(x,y)\eta_i(x,y)$, $\quad \bar{\varsigma}(x,y) = S_{I_1,I_2}(x,y)\bar{\eta}(x,y)$, and $\eta_i^*(x,y), \bar{\eta}^*(x,y), \varsigma_i^*(x,y), \bar{\varsigma}^*(x,y)$ be their bootstrap counterparts.

Theorem 2 *The processes $\{n^{1/2}\bar{\eta}(x,y)\}$ and $\{n^{1/2}[\bar{\eta}^*(x,y)-\bar{\eta}(x,y)]\}$ both converge weakly on $D[0,S]\times D[0,T]$ to a mean zero Gaussian process $Z(x,y)$ with covariance structure $Cov(Z(x',y'), Z(x,y)) = \Gamma((x',y'),(x,y))$.*

Proof Let $\bar{A}(x) = (1/n)\sum_{i=1}^{n} A(X_i, \xi(\vec{X}_i), x)$ and

$$\bar{W}(x,y) = (1/n)\sum_{i=1}^{n}[H(x,0)]^{-1}A_x(Y_i, \xi(\vec{Y}_i), y)I(X_i > x),$$

so that $\bar{\eta}(x,y) = \bar{A}(x) + \bar{W}(x,y)$. Let $H_n(x,y) = (1/n)\sum_{i=1}^{n} I(X_i > x, Y_i > y)$ and

$$H_{1n}(x,y) = (1/n)\sum_{i=1}^{n} I(X_i > x, Y_i > y, \xi(\vec{Y}_i) \in \varphi_{I_2})$$

be the empirical survival function and subsurvival function. From (4), we have

$$\begin{aligned}
\bar{W}(x,y) &= -\frac{1}{n}\sum_{i=1}^{n}[\int_0^{Y_i\wedge y} H^{-2}(x,v)dH_1(x,v) \\
&\quad + H^{-1}(x,Y_i)I(Y_i \le y, \xi(\vec{Y}_i) \in \varphi_{I_2})]I(X_i \ge x) \\
&= -\int_0^y H^{-2}(x,v)H_n(x,v)dH_1(x,v) + \int_0^y H^{-1}(x,v)dH_{1n}(x,v) \\
&= -\int_0^y H^{-2}(x,v)[H_n(x,v) - H(x,v)]dH_1(x,v) \\
&\quad + \int_0^y H^{-1}(x,v)d[H_{1n}(x,v) - H_1(x,v)] \\
&= -\int_0^y H^{-2}(x,v)[H_n(x,v) - H(x,v)]dH_1(x,v) \\
&\quad + [H_{1n}(x,y) - H_1(x,y)]H^{-1}(x,y) \\
&\quad - \int_0^y [H_{1n}(x,v) - H_1(x,v)]dH^{-1}(x,v)
\end{aligned}$$

and this is just $n^{-1/2}[A_{2n}(\vec{t}) + B_{2n}(\vec{t})]$ on page 252 of [7], where $\vec{t} = (x,y)$.

Similary, one can show that $\bar{A}(x,y) = n^{-1/2}[A_{1n}(\vec{t}) + B_{1n}(\vec{t})]$ as defined on page 251 of [7]. The weak convergence of $n^{1/2}\bar{\eta}(x,y)$ now follows from the proof of Theorem 1 of [7]. The weak convergence of the bootstrap process $n^{1/2}[\bar{\eta}^*(x,y) - \bar{\eta}(x,y)]$ to a Gaussian process follows from similar arguments as in Sections 3 and 4 of [7]. To show that it converges to the same Gaussian process $Z(x,y)$ one only need to check that the finite dimensional distribution $n^{1/2}[\bar{\eta}^*(x_i,y_i)-\bar{\eta}(x_i,y_i)]$, for some $\{(x_i,y_i), 1\le i\le k\}$, converges to the k-variate Gaussian distribution with mean zero and covariance matrix $(\Gamma((x_i,y_i),(x_j,y_j)))$. This follows directly from the bootstrap central limit theorem for sample means [8]. The theorem is thus completed.

Corollary 2 *The processes $n^{1/2}(\hat{S}_{I_1,I_2}(x,y)-S_{I_1,I_2}(x,y))$ and $n^{1/2}(\hat{S}^*_{I_1,I_2}(x,y)-\hat{S}_{I_1,I_2}(x,y))$ both converge to the two-parameter Gaussian process with mean zero and covariance*

$$S_{I_1,I_2}(x,y)S_{I_1,I_2}(x',y')\Gamma((x,y),(x',y')).$$

We have thus shown that the bootstrap method works under the bivariate competing risks model, which provides a way to estimate the standard error of $\hat{S}_{I_1,I_2}(x,y)$ or to construct a confidence band for $S_{I_1,I_2}(x,y)$. This is valuable since the covariance structure of $\hat{S}_{I_1,I_2}(x,y)$ is very complicated as shown in Proposition 1.

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二元竞争风险场合二元 PL 估计的弱收敛性及其 Bootstrap 结果

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摘　要

本文考虑了在二元竞争风险场合下二元 PL 估计的问题．文中给出了关于二元 PL 估计的独立同分布表达式，即将估计的差表为均值为零的独立同分布过程的和，且其余项的阶为 $O((n^{-1}\log n)^{3/4})$ a. s.，并证明该过程弱收敛到一个二元 Gauss 过程．关于 Bootstrap 情况也得到了类似的结果．