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A Necessary and Sufficient Condition on Convergence of Bounded Linear Operators in the Space of Continuous Functions*

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Abstract. Korovkin [1] gave a necessary and sufficient condition on linear positive approximation in $C[a,b]$. In this note, we give such a condition on general bounded approximation in $C[a,b]$

Let B be the space of bounded linear operators on $C[a,b]$ to $C[a,b]$. For convenience, we denote $L(f)$ by $L(f,x)$ or $L(f(t),x)$, where $L \in B, f \in C[a,b]$.

Theorem *Let $\{L_n\} \subset B$. $L_n(f,x) \to f(x)$ $(n \to \infty)$ uniformly on $x \in [a,b]$ for all $f \in C[a,b]$ if and only if:*

(i) There exists $\{\delta_n\} \subset B$ such that $\delta_n(f,x) \to 0$ $(n \to \infty)$ uniformly on $x \in [a,b]$ for every $f \in C[a,b]$, and for any $f \in C[a,b]$, $f(t) \geq 0, t \in [a,b]$,

$$L_n(f,x) + \delta_n(f,x) \geq 0 \tag{1}$$

is valid for $n = 1,2,\cdots, x \in [a,b]$.

(ii) $L_n(f_i,x) \to f_i(x)$ $(n \to \infty)$ uniformly on $x \in [a,b]$ for $f_i(t) = t^i, i = 0,1,2$.

Proof For the necessity, we prove only (i). Let I be the identity element in B. Then $I - L_n \in B$. Since $(I - L_n)(f,x) = f(x) - L_n(f,x) \to 0$ uniformly on $x \in [a,b]$ for any $f \in C[a,b]$ $(n \to \infty)$, and $L_n(f,x) + (I - L_n)(f,x) = f(x) \geq 0$ for $n = 1,2,\cdots, x \in [a,b]$. If $f \in C[a,b], f(t) \geq 0, t \in [a,b]$, (i) is true for $\delta_n = I - L_n, n = 1,2,\cdots,$

On the contrary, let (i) and (ii) be valid for L_n, δ_n. For any $f \in C[a,b]$, since it is continuous uniformly on $[a,b]$, then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$-\left(\frac{2\parallel f \parallel}{\delta^2}(t-x)^2 + \epsilon\right) \leq f(t) - f(x) \leq \frac{2\parallel f \parallel}{\delta^2}(t-x)^2 + \epsilon, t, x \in [a,b].$$

Let $F_x^{(1)}(t) = f(t) - f(x) + \frac{2\|f\|}{\delta^2}(t-x)^2 + \epsilon$. Then for any fixed $x \in [a,b]$, one has $F_x^{(1)} \geq 0$ $(t \in [a,b])$, hence $L_n(F_x^{(1)}(t), y) + \delta_n(F_x^{(1)}(t), y) \geq 0, n = 1,2,\cdots, y \in [a,b]$.

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Naturally, $L_n(F_x^{(1)}(t),x)+\delta_n(F_x^{(1)}(t),x)\geq 0, n=1,2,\cdots$. By linearity of δ_n we have

$$
\begin{aligned}
& |\delta_n(F_x^{(1)}(t),x)| \\
= \ & |\delta_n(f(t),x)-f(x)\delta_n(1,x)+\frac{2\parallel f\parallel}{\delta^2}[\delta_n(t^2,x)-2x\delta_n(t,x)+x^2\delta_n(1,x)]+\epsilon\delta_n(1,x)| \\
\leq \ & |\delta_n(f(t),x)|+\parallel f\parallel |\delta_n(1,x)|+\frac{2\parallel f\parallel}{\delta^2}[|\delta_n(t^2,x)| \\
+ \ & 2C|\delta_n(t,x)|+C^2|\delta_n(1,x)|]+\epsilon|\delta_n(1,x)|, \qquad (2)
\end{aligned}
$$

where $C=\max\{|a|,|b|\}$. By (2) and (i) we know that $\delta_n(F_x^{(1)}(t),x)\to 0 \quad (n\to\infty)$ uniformly on $x\in[a,b]$. Then

$$
\begin{aligned}
& L_n(f(t),x)-L_n(f(x),x) \\
\geq \ & -\frac{2\parallel f\parallel}{\delta^2}L_n((t-x)^2,x)-\epsilon L_n(1,x)-\delta_n(F_x^{(1)}(t),x). \qquad (3)
\end{aligned}
$$

By the same discussion we can get

$$
\begin{aligned}
& L_n(f(t),x)-L_n(f(x),x) \\
\leq \ & \frac{2\parallel f\parallel}{\delta^2}L_n((t-x)^2,x)+\epsilon L_n(1,x)+\delta_n(F_x^{(2)}(t),x), \qquad (4)
\end{aligned}
$$

where $F_x^{(2)}(t)=-f(t)+f(x)+\frac{2\|f\|}{\delta^2}(t-x)^2+\epsilon$, and $\delta_n(F_x^{(2)}(t),x)\to 0 \ (n\to\infty)$ uniformly on $x\in[a,b]$.

Notice that $L_n(f(x),x)=f(x)L_n(1,x)\to f(x), L_n((t-x)^2,x)\to 0$ uniformly on $x\in[a,b]$ when $n\to\infty$, the desired result is achived by (3) and (4). The proof is complete.

Reference

[1] P.P.Korovkin, *On convergence of linear positive operators in the space of continuous of functions*, Doklady, S.S.S.R., **90**(1953), 961–964.